On Tracking a Deterministic Growth

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摘要

本論文運用隨機綫性二次控制理論研究某些連續時間的追蹤模型。該類模型的投資目標以最小化投資組合回報值與某個固定增長值在整個投資時間區間里的差異的方式，去追蹤該連續增長值。我們將會討論在兩種不同的市場環境下的模型。一種是所有的市場參數，包括銀行利率和股票回報及波動率，都是時間的確定性函數；另一種是這些參數在若干個有限的狀態之間隨機變換，並用馬爾科夫鏈來描述。在每一種市場條件下，我們考慮不同的模型，它們的區別在於有無時間終端之期望回報的限制條件。無此限制的問題可以直接用一般的綫性二次控制方法解決；而有此限制的問題則運用拉格朗日乘子方法轉換為無限制的最優控制問題，然後用對偶定理來解決。

Abstract

In this thesis, some continuous-time tracking models are proposed and solved by the general stochastic linear-quadratic (LQ) approach. The objective is to make the investment result close to a given continuously compounding growth, in the sense of minimizing the expected total difference between the portfolio value and the growth wealth over an investment horizon. We study the models in two different market environments. One is when all the market parameters, including the bank interest rate and the appreciation and volatility rates of the stocks, are deterministic functions of time, and the other is when all these parameters switch among a finite number of states, modulated by a Markov chain. In each type of markets, we study models with or without a terminal return constraint. The problem without terminal return constraint is solved directly by using the general LQ method, whereas the problem with terminal return constraint is reduced to an unconstrained problem via the Lagrange multiplier technique and then solved by a dual theorem.

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Chapter 1

Introduction and Literature Review

Portfolio selection is concerned with the allocation of wealth among a basket of securities. The mean-variance model was originally proposed by Markowitz [16] [17] for portfolio construction in a single period, so as to achieve a prescribed tradeoff between the return of the investment and the associated risk. The most important contribution of Markowitz's work is the introduction of quantitative and scientific approaches to risk management and analysis. In his model, the risk is quantified by using the variance, which enables investors to seek the highest return after specifying their acceptable risk level. This work has become the foundation of modern finance theory and has had tremendous impact on its further development.

After Markowitz's pioneering work, the mean-variance model was extended to dynamic mean-variance models. Along this line, multi-period mean-variance portfolio selection was studied in, for example, Mossin [18], Samuelson [20], Hakansson [10], Grauer and Hakansson [9], and Pliska [19] among others. Specifically, rather than treating Var $x(T)$ and $Ex(T)$ (where $x(T)$ is the terminal wealth) of a portfolio as separate quantities and finding the relationship between them, the expected utility of terminal wealth $EU(x(T))$ is considered instead. The conflicting *profit-seeking yet risk-averse* nature of the investor is captured by the utility function U, which is usually a power, log, exponential, or quadratic form. A disadvantage of this approach is that the relationship between risk and return is contained only implicitly in the utility function. Hence, it is less clear in general which relationship exists between the risk and the return of the derived policy. It should be noted that mean-variance analysis and expected utility approach are two different tools in dealing with portfolio selection, accounting for different degrees of risk aversion. On the other hand, related mean-variance hedging problems were attacked by Schweizer [21] for discrete-time models and Duffie and Richardson [7] for continuous-time models where dynamic strategies were derived, based on the projection theorem, to hedge contingent claims in incomplete markets with two assets. In particular, in [7], the result was derived under the assumption that all the coefficients (interest rate, volatility rate, etc.) are deterministic, time-invariant constants.

It is difficult to completely mimic Markowitz's idea in the multi-period or continuous-time settings, because the variance Var $x(T)$ involves a term $[Ex(T)]^2$ that is hard to analyze due to its non-separability in the sense of dynamic programming; see [24] for a more detailed discussion. Recently Li and Ng [12] faithfully extended Markowitz's mean-variance model to the multi-period setting by using the idea of embedding the problem into a tractable auxiliary problem. Moreover, in the paper of Zhou and Li [24], the continuous-time mean-variance problem is formulated as a stochastic linear-quadratic (LQ) optimal control problem. The solution to this problem is obtained using the embedding technique introduced in Li and Ng [12] and results from stochastic LQ control. It is important to recognize that the main contribution of [24] is not that the explicit mean-variance efficient frontier it obtained; rather it is the introduction of the unifying framework, which can solve certain finance problems including mean-variance portfolio selection. Such an approach bridges portfolio selection problems and standard stochastic control models.

There are many advantages of using the stochastic control to study dynamic portfolio selection problems. First, the theory of stochastic control is so rich that many mathematical machineries are available; see Fleming and Soner [8] and Yong and Zhou [23]. Secondly, a unified study of various mean-variance type problems in finance can be undertaken in this framework. For example, a mean-variance hedging problem was treated in [11] within the LQ framework, a portfolio selection problem with short sell prohibition was studied in [13] via LQ and viscosity solution theories, a problem with random coefficients was solved in [14] using LQ theory and backward stochastic differential equations (BSDE), and a problem with regime switching was handled in [25] as a Markov-modulated stochastic LQ control model. Finally, the LQ framework provides an opportunity to deal with more complicated finance problems that are linear-quadratic in nature.

In this thesis, we consider an investor whose investment objective is to track a given continuously compounding growth over a finite time horizon, in the way of controlling the deviation from the growth both during the process and at the terminal time, depending on whether or not a terminal return constraint is present. The reason why we have a tracking term is to get a steady growth, thus reducing the possibility of bankruptcy. For example, we consider a fund manager who will give some intermediate reports to his customers during the investment time period. With this kind of investment strategy, he can show his clients a tendency of wealth growth. The purpose of this investment is different from that of the mean-variance model, the latter being only concerned with the relationship between terminal return and risk. However, we can still use a LQ control framework to solve our models.

The analysis of the this thesis is for markets consisting of one bank account and multiple stocks. We will study the tracking models in two different market environments. One is a market whose key parameters, such as the bank interest rate, stocks appreciation rates, and volatility rates, are all deterministic functions of time. The other one is the so-called regime switching market, whose market parameters depend on the market modes that switch between a finite number of states. Here the random switching is assumed to be independent of the random sources that drive the stock prices. This is motivated by the need for formulating more realistic models that better reflect random market environment. A regime switching model can be mathematically formulated as a stochastic differential equation whose coefficients are modulated by a continuous-time Markov chain. Such type of models have been mainly employed in literature to discuss options, see e.g., [1], [3], [6] and [22].

The rest of the thesis is arranged as follows. Chapter 2 begins with the formulation of models under consideration. Chapter 3 is concerned with the solutions in the market with deterministic parameters. In Chapter 4, the models with regime switching are solved via Markov-modulated stochastic LQ control. Finally, Chapter 5 concludes the thesis.

Chapter 2

The Tracking Portfolio Models

In this chapter, we give the general formulation and assumptions of the models under consideration. A basic approach to solving the models is introduced in the last section of this chapter.

2.1 Problem Formulation

Throughout the thesis, unless otherwise specified, T is a fixed terminal time and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which defined a standard $\mathcal{F}_t$-adapted d-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^d(t))'$ with $W(0) = 0$. Specific assumptions on this basic framework will be made in individual chapters.

Notation. We use the following notation:

$\mathbb{R}^d$: the space of d-dimensional column vectors;

M': the transpose of a vector or matrix M;

M^j: the j-th entry of any vector M;

$tr(M)$: the trace of a square matrix M;

S^n: the space of all $n \times n$ symmetric matrices;

S^n_+: the subspace of all nonnegative definite matrices of S^n;

$C(0, T; \mathbb{R}^d)$: the Banach space of $\mathbb{R}^d$-valued continuous function on $[0, T]$ endowed with the maximum norm;

$L^2_{\mathcal{F}_t}(0,T;\mathbb{R}^d)$: the set of all $\mathbb{R}^d$-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |f(t)|^2 dt < +\infty$.

Suppose there is a market in which $d+1$ *assets* are traded continuously. One of the assets is *a bank account* whose price process $p_0(t)$ is subject to the following (stochastic) ordinary differential equation (ODE):

$$\begin{cases} dp_0(t) = r(t)p_0(t)dt, \ t \in [0,T], \\ p_0(0) = p_0 > 0, \end{cases} \tag{2.1}$$

where $r(t) \geq 0$ is the *interest rate*. The other d assets are *stocks* whose price processes $p_1(t), \cdots, p_d(t)$ satisfy the following stochastic differential equation (SDE):

$$\begin{cases} dp_m(t) = p_m(t)\{b_m(t)dt + \sum_{n=1}^d \sigma_{mn}(t)dW^n(t)\}, \ t \in [0,T], \\ p_m(0) = p_m > 0, \ m = 1, 2, \cdots, d, \end{cases} \tag{2.2}$$

where $b_m(t) > 0$ is the *appreciation rate*, and $\sigma_m(t) \equiv (\sigma_{m1}(t), \cdots, \sigma_{md}(t))$ is the *volatility* or the *dispersion rate* of the m^{th} stock. We assume that $r(t)$, $b_m(t)$, $\sigma_{mn}(t)$ are scalar-valued, $\mathcal{F}_t$-adapted and uniformly bounded stochastic processes.

Define the *volatility matrix* $\sigma(t) := (\sigma_1(t), \cdots, \sigma_d(t))' = (\sigma_{mn}(t))_{d\times d}$. The basic assumption throughout this thesis is

$$\sigma(t)\sigma(t)' \geq \delta I, \ \forall t \in [0,T], \ P - a.s. \tag{2.3}$$

for some $\delta > 0$. This is the so-called non-degeneracy condition.

Consider an agent whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assuming that the trading of shares takes place continuously and transaction cost and consumptions are not considered, then $x(\cdot)$ satisfies (see, e.g. [23], p57)

$$\begin{cases} dx(t) &= \{r(t)x(t) + \sum_{m=1}^{d}[b_m(t) - r(t)]\pi_m(t)\}dt \\ &\quad + \sum_{n=1}^{d}\sum_{m=1}^{d}\sigma_{mn}(t)\pi_m(t)dW^n(t) \\ x(0) &= x_0 > 0, \end{cases} \tag{2.4}$$

where $\pi_m(t)$, $m = 0, 1, 2, \cdots, d$, denotes the total market value of the agent's wealth in the m^{th} asset at time t. We call $\pi(\cdot) := (\pi_1(\cdot), \cdots, \pi_d(\cdot))'$ a *portfolio* of the agent.

Note that once $\pi(\cdot)$ is determined, $\pi_0(\cdot)$, the wealth in the bank account is completely specified by $\pi_0(t) = x(t) - \sum_{m=1}^{d}\pi_m(t)$. Thus, in our following analysis, only $\pi(\cdot)$ is considered.

Setting

$$B(t) := (b_1(t) - r(t), \cdots, b_d(t) - r(t)), \tag{2.5}$$

then, equation (2.4) can be rewritten as

$$\begin{cases} dx(t) &= [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) &= x_0. \end{cases} \tag{2.6}$$

Definition 2.1 A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}^d)$, and the SDE (2.6) has a unique solution $x(\cdot)$ corresponding to $\pi(\cdot)$. In this case, we refer to $(x(\cdot), \pi(\cdot))$ as an *admissible (wealth-portfolio) pair.*

The agent's objective is to find an admissible portfolio that tracks a given growth function most closely. In this thesis, the target of tracking is an exponential function with parameter μ.

We now define the models to be investigated.

Definition 2.2 Model I refers to the following optimization problem:

$$\begin{cases} \text{Minimize} \;\; J_1(x_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T [x(t) - e^{\mu t} x_0]^2 dt + \frac{1-\xi}{2} E[x(T) - e^{\mu T} x_0]^2 \\ \text{subject to} \;\; (x(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \tag{2.7}$$

where $\mu > 0$ is a given growth rate and $0 \leq \xi \leq 1$.

Note that in this model the objective function includes both the running and terminal terms, and ξ is the weight between the two.

In Model I, the portfolio only concerns the tracking of the growth return, and not the terminal return. Model II incorporates the latter into the formulation.

Definition 2.3 Model II refers to the following optimization problem:

$$\begin{cases} \text{Minimize} \;\; J_2(x_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T [x(t) - e^{\mu t} x_0]^2 dt + \frac{1-\xi}{2} \text{ Var } x(T) \\ \text{subject to} \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \end{cases} \tag{2.8}$$

where $\mu > 0$ is a deterministic growth rate, $0 \leq \xi < 1$, and $z \in \mathbb{R}$ is given.

The above model is a constrained stochastic optimization problem. The problem is called *feasible* if there is at least one portfolio satisfying all the constraints. The problem is called *finite*, if it is feasible and the infimum of $J_2(x_0, \pi(\cdot))$ is finite. Finally, an optimal portfolio to Model II, if it ever exists, is called a(n) *(tracking) efficient portfolio* corresponding to z, the corresponding $(\text{Var } x(T), z) \in \mathbb{R}^2$ and $(\sigma_{x(T)}, z) \in \mathbb{R}^2$ar e interchangeably called a(n) *(tracking) efficient point*, where $\sigma_{x(T)}$ denotes the standard deviation of $x(T)$. The set of all the efficient points is called the *(tracking) efficient frontier* which is different from the mean-variance efficient frontier.

The case when $\xi = 1$ in Model II is separated as Model III.

Definition 2.4 Model III refers to the following optimization problem:

$$\begin{cases} \text{Minimize} \;\; J_3(x_0, \pi(\cdot)) = \frac{1}{2} E \int_0^T [x(t) - e^{\mu t} x_0]^2 dt \\ \text{subject to} \;\; \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \end{cases} \tag{2.9}$$

where $\mu > 0$ is a deterministic growth rate and $z \in \mathbb{R}$ is given.

2.2 Reformulation of Tracking Models

In this section we reformulate Models I-III so as to fit the general stochastic LQ framework.

Let $y(t) = x(t) - e^{\mu t} x_0$, $f(t) = (r(t) - \mu) x_0 e^{\mu t}$. If $(x(\cdot), \pi(\cdot))$ is an admissible pair, then $(y(\cdot), \pi(\cdot))$ satisfies

$$\begin{cases} dy(t) = [r(t)y(t) + B(t)\pi(t) + f(t)]dt + \pi'(t)\sigma(t)dW(t) \\ y(0) = y_0 = 0 \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d). \end{cases} \tag{2.10}$$

We call $(y(\cdot), \pi(\cdot))$ an admissible pair if it satisfies (2.10).

Clearly, Model I is equivalent to

$$\begin{cases} \text{Minimize } \; J_1(y_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T y^2(t)dt + \frac{1-\xi}{2} E[y^2(T)], \; \xi \in [0,1] \\ \text{subject to } (y(\cdot), \pi(\cdot)) \text{ admissible.} \end{cases} \tag{2.11}$$

Notice that if $Ex(T) = z$, then $Ey(T) = z - e^{\mu T} x_0 := \hat{z}$. Moreover, $\text{Var } y(T) = \text{Var } x(T) = E[x(T) - z]^2 = E[y(T) - \hat{z}]^2$. Hence, Model II is equivalent to

$$\begin{cases} \text{Minimize } \; J_2(y_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T y^2(t)dt + \frac{1-\xi}{2} E[y(T) - \hat{z}]^2, \; \xi \in [0,1) \\ \text{subject to } \begin{cases} Ey(T) = \hat{z} \\ (y(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \end{cases} \tag{2.12}$$

and Model III is equivalent to

$$\begin{cases} \text{Minimize } \; J_3(y_0, \pi(\cdot)) = \frac{1}{2} E \int_0^T y^2(t)dt \\ \text{subject to } \begin{cases} Ey(T) = \hat{z} \\ (y(\cdot), \pi(\cdot)) \text{ admissible.} \end{cases} \end{cases} \tag{2.13}$$

2.3 A Stochastic LQ Control Approach

The tracking problems formulated in the previous sections will be solved via a stochastic LQ optimal control technique. To solve model (2.11), we can use the general solution in [24]. In this section, we outline the approach to solving (2.12) and (2.13).

The model (2.12) is a dynamic optimization problem with a constraint $Ey(T) = \hat{z}$. To handle this constraint, we apply the Lagrange multiplier technique. Define

$$\begin{aligned}\tilde{J}_2(y_0, \pi(\cdot); \lambda) &:= \tfrac{\xi}{2} E \int_0^T y^2(t)dt + \tfrac{1-\xi}{2}\{E[y(T) - \hat{z}]^2 + 2\lambda(y(T) - \hat{z})\} \\ &= \tfrac{\xi}{2} E \int_0^T y^2(t)dt + \tfrac{1-\xi}{2}\{E[y(T) + \lambda - \hat{z}]^2 - \lambda^2\},\ \xi \in [0,1).\end{aligned}$$

Since $J_2(y_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$ and the constraint function $Ey(T) = \hat{z}$is affine in $\pi(\cdot)$, we can apply the well-known duality theorem (see, e.g. [15]) to conclude that if problem (2.12) is finite for every $\hat{z} \in \mathbb{R}$, then the optimal value is

$$\begin{aligned}J_2(y_0, \pi^*(\cdot)) &:= \inf_{\text{all constraints}} J_2(y_0, \pi(\cdot)) \\ &= \sup_{\lambda \in \mathbb{R}} \inf_{\pi(\cdot)\ \text{all constraints}} \tilde{J}_2(y_0, \pi(\cdot); \lambda) > -\infty.\end{aligned} \tag{2.14}$$

Based on this, we can solve the original problem (2.12) in the following three steps (see [14]):

Step 1. For each fixed λ, solve the problem

$$\begin{cases} \min\ \ \tilde{J}_2(y_0, \pi(\cdot); \lambda) = \tfrac{\xi}{2} E \int_0^T y^2(t)dt + \tfrac{1-\xi}{2}\{E[y(T) + \lambda - \hat{z}]^2 - \lambda^2\} \\ \text{subject to}\ \ (y(\cdot), \pi(\cdot)) \text{ admissible.} \end{cases} \tag{2.15}$$

This is a special case of a more general class of problems, namely the indefinite stochastic LQ control problems. Solving this problem leads to the optimal portfolio $\pi^*_\lambda(\cdot)$ and the optimal value $\tilde{J}_2(y_0, \pi^*_\lambda(\cdot); \lambda)$.

Step 2. Solve the problem $\sup_{\lambda \in R} \tilde{J}_2(y_0, \pi_\lambda^*(\cdot); \lambda)$ to get the unique optimal λ^* and the corresponding value $\tilde{J}_2(y_0, \pi_{\lambda^*}^*(\cdot); \lambda^*)$.

Step 3. The unique optimal portfolio for (2.12) is then $\pi_{\lambda^*}^*(\cdot)$, and the optimal value is $J_2(y_0, \pi^*(\cdot)) := \tilde{J}_2(y_0, \pi_{\lambda^*}^*(\cdot); \lambda^*)$.

Remark 2.1 Comparing (2.11) to (2.15), we see that $J_1(y_0, \pi(\cdot))$ is a special case of $\tilde{J}_2(y_0, \pi(\cdot); \lambda)$ with $\lambda = 0$ and $z = x_0 e^{\mu T}$, when $0 \leq \xi < 1$.

Finally, we can use the same three-step method to solve (2.13). The only difference is to change $\tilde{J}_2(y_0, \pi_\lambda^*(\cdot); \lambda)$ to $\tilde{J}_3(y_0, \pi_{\hat{\lambda}}^*(\cdot); \hat{\lambda})$, and consider the following

$$\begin{cases} \text{Minimize} \quad \tilde{J}_3(y_0, \pi(\cdot); \hat{\lambda}) = \frac{1}{2} E \int_0^T y^2(t) dt + \hat{\lambda} E(y(T) - \hat{z}) \\ \text{subject to} \quad (y(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \tag{2.16}$$

and $J_3(y_0, \pi^*(\cdot)) := \tilde{J}_3(y_0, \pi_{\hat{\lambda}^*}^*(\cdot); \hat{\lambda}^*)$.

Chapter 3

Efficient Tracking: Deterministic Market Parameters

In this chapter, we assume that all the market parameters are deterministic. This means that $r(t)$, $b_m(t)$ and $\sigma_{mn}(t)$ are deterministic functions of time t.

Denote

$$\rho(t) = B(t)[\sigma(t)\sigma(t)']^{-1}B(t)'. \tag{3.1}$$

The following is about the feasibility of Model II and III (see [25], Theorem 3.1).

Proposition 3.1 *The problem (2.8) (or (2.9)) is feasible for every $z \in \mathbb{R}$ if and only if*

$$E\int_0^T |B(t)|^2 dt > 0. \tag{3.2}$$

The condition (3.2) holds as long as there is one stock whose appreciation rate is different from the interest rate, which is a practically reasonable assumption.

3.1 Solution to Model I

We introduce the following equations:

$$\begin{cases} \dot{P}(t) = [\rho(t) - 2r(t)]P(t) - \xi \\ P(T) = 1 - \xi, \end{cases} \tag{3.3}$$

along with an equation

$$\begin{cases} \dot{R}(t) = [r(t) + \frac{\xi}{P(t)}]R(t) - f(t) \\ R(T) = 0. \end{cases} \tag{3.4}$$

Lemma 3.1 *The solutions of (3.3) must satisfy*

(a) *If* $0 \leq \xi < 1$, *then* $P(t) > 0,\ \forall\, t \in [0, T]$.

(b) *If* $\xi = 1$, *then* $P(t) > 0,\ \forall\, t \in [0, T)$.

Proof. We write the solution of (3.3) as

$$P(t) = (1-\xi)e^{-\int_t^T [\rho(s)-2r(s)]ds} + \xi \int_t^T e^{-\int_t^\tau [\rho(s)-2r(s)]ds} d\tau. \tag{3.5}$$

Since $e^{-\int_t^T [\rho(s)-2r(s)]ds} > 0$, and $\int_t^T e^{-\int_t^\tau [\rho(s)-2r(s)]ds} d\tau > 0$, we get the results of (a) and (b) immediately. □

Proposition 3.2 *Problem (2.11) has an optimal feedback control*

$$\pi_1^*(t, y) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[y + R(t)], \tag{3.6}$$

and the corresponding optimal value is

$$J_1(y_0, \pi_1^*(\cdot)) = \int_0^T P(t)R(t)[f(t) - \frac{1}{2}\rho(t)R(t)]dt. \tag{3.7}$$

Proof. Let $(y(\cdot), \pi(\cdot))$ be an admissible pair. Applying Ito's formula, we get

$$
\begin{aligned}
& \tfrac{1}{2}d\{P(t)y^2(t)\} \\
= \; & \tfrac{1}{2}\{P_t(t)y^2(t)dt + 2P(t)y(t)dy + P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)dt + \{\cdots\}dW(t) \\
= \; & \tfrac{1}{2}\{[(\rho(t) - 2r(t))P(t) - \xi]y^2(t) + 2P(t)y(t)[r(t)y(t) + B(t)\pi(t) + f(t)] \\
& +P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\}dt + \{\cdots\}dW(t) \\
= \; & \tfrac{1}{2}\{(\rho(t)P(t) - \xi)y^2(t) + 2P(t)y(t)[B(t)\pi(t) + f(t)] \\
& +P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\}dt + \{\cdots\}dW(t),
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
& d\{P(t)R(t)y(t)\} \\
= \; & \{[(\rho(t) - 2r(t))P(t) - \xi]R(t)y(t) + [(r(t)P(t) + \xi)R(t) - f(t)P(t)]y(t) \\
& +P(t)R(t)[r(t)y(t) + B(t)\pi(t) + f(t)]\}dt + \{\cdots\}dW(t) \\
= \; & \{[\rho(t)R(t) - f(t)]P(t)y(t) + P(t)R(t)[B(t)\pi(t) + f(t)]\}dt \\
& +\{\cdots\}dW(t).
\end{aligned} \tag{3.9}
$$

Integrating both (3.8) and (3.9) from 0 to T, taking expectations, adding them together, we obtain

$\frac{1}{2}(1-\xi)Ey^2(T)$

$$
\begin{aligned}
&= E\int_0^T\{\tfrac{1}{2}\{[(\rho(t)-2r(t))P(t)-\xi]y^2(t) \\
&\quad +2P(t)y(t)[r(t)y(t)+B(t)\pi(t)+f(t)]+P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\} \\
&\quad +\{[\rho(t)R(t)-f(t)]P(t)y(t)+P(t)R(t)[B(t)\pi(t)+f(t)]\}dt \\
&= E\int_0^T\{\tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)+P(t)[y(t)+R(t)]B(t)\pi(t) \\
&\quad +\tfrac{1}{2}[\rho(t)-\xi]y^2(t)+\rho(t)P(t)R(t)y(t)+f(t)P(t)R(t)\}dt \\
&= E\int_0^T\{\tfrac{1}{2}P(t)\left[\pi(t)+[\sigma(t)\sigma(t)']^{-1}B(t)'[y+R(t)]\right]' \\
&\quad [\sigma(t)\sigma(t)']\left[\pi(t)+[\sigma(t)\sigma(t)']^{-1}B(t)'[y+R(t)]\right] \\
&\quad +f(t)P(t)R(t)-\tfrac{\xi}{2}y^2(t)-\tfrac{1}{2}\rho(t)P(t)R^2(t)\}dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
J_1(y_0,\pi(\cdot)) &= E\int_0^T\{\tfrac{1}{2}P(t)\left[\pi(t)+[\sigma(t)\sigma(t)']^{-1}B(t)'[y+R(t)]\right]' \\
&\quad [\sigma(t)\sigma(t)']\left[\pi(t)+[\sigma(t)\sigma(t)']^{-1}B(t)'[y+R(t)]\right] \\
&\quad +f(t)P(t)R(t)-\tfrac{1}{2}\rho(t)P(t)R^2(t)\}dt.
\end{aligned}
$$

Since $P(t)>0$ by Lemma 3.1, it follows immediately that the optimal feedback control is given by (3.6) and the optimal value is given by (3.7), provided that the corresponding equation (2.10) under the feedback control (3.6) has a solution. However, under (3.6), the system (2.10) is a nonhomogeneous linear stochastic differential equation. Since all the coefficients of this linear equation are uniformly bounded, the existence and uniqueness of the solution to the equation are straightforward based on a standard successive approximation scheme. □

Theorem 3.1 *Problem (2.7) has an optimal feedback control*

$$\pi_1^*(t,x) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[x - x_0e^{\mu t} + R(t)], \tag{3.10}$$

and the corresponding optimal value is

$$J_1(x_0, \pi_1^*(\cdot)) = \int_0^T P(t)R(t)[f(t) - \frac{1}{2}\rho(t)R(t)]dt. \tag{3.11}$$

Proof. Since $y = x - x_0e^{\mu t}$, we get the result immediately from Proposition 3.2. □

Next, we proceed to derive the terminal return and variance under the optimal feedback control (3.10).

Denote $\varphi(t) = x_0e^{\mu t} - R(t) = x_0e^{\mu t} - \int_t^T f(\tau)e^{-\int_t^\tau [r(s)+\frac{\xi}{P(s)}]ds}d\tau$. Then (3.10) reduces to

$$\pi_1^*(t,x) = [\sigma(t)\sigma(t)']^{-1}B(t)'(\varphi(t) - x). \tag{3.12}$$

Under the optimal feedback control (3.12) the wealth equation (2.6) evolves as

$$\begin{cases} dx(t) &= [(r(t)-\rho(t))x(t)+\rho(t)\varphi(t)]dt \\ &\quad +B(t)[\sigma(t)\sigma(t)']^{-1}\sigma(t)[\varphi(t)-x(t)]dW(t) \\ x(0) &= x_0 > 0. \end{cases} \tag{3.13}$$

Applying Ito's formula to $Ex(t)$ and $Ex^2(t)$, we obtain

$$\begin{cases} dEx(t) = [(r(t)-\rho(t))Ex(t)+\rho(t)\varphi(t)]dt \\ Ex(0) = x_0 > 0, \end{cases} \tag{3.14}$$

and

$$\begin{cases} dEx^2(t) = [(2r(t)-\rho(t))Ex^2(t)+\rho(t)\varphi^2(t)]dt \\ Ex^2(0) = x_0^2 > 0. \end{cases} \tag{3.15}$$

Solving (3.14) and (3.15), we can express $Ex(T)$ and $Ex^2(T)$ as

$$Ex(T) = x_0 e^{\int_0^T [r(s)-\rho(s)]ds} + \int_0^T \varphi(t)\rho(t)e^{\int_t^T [r(s)-\rho(s)]ds}dt, \tag{3.16}$$

$$Ex^2(T) = x_0^2 e^{\int_0^T [2r(s)-\rho(s)]ds} + \int_0^T \varphi^2(t)\rho(t)e^{\int_t^T [2r(s)-\rho(s)]ds}dt. \tag{3.17}$$

Moreover,

$$\begin{aligned} \text{Var } x(T) &= Ex^2(T) - [Ex(T)]^2 \\ &= x_0^2 e^{\int_0^T [2r(s)-\rho(s)]ds} + \int_0^T \varphi^2(t)\rho(t)e^{\int_t^T [2r(s)-\rho(s)]ds}dt \\ &\quad - \left[x_0 e^{\int_0^T [r(s)-\rho(s)]ds} + \int_0^T \varphi(t)\rho(t)e^{\int_t^T [r(s)-\rho(s)]ds}dt\right]^2. \end{aligned} \tag{3.18}$$

3.2 A Special Case of Model I

In this section, we consider a special case of Model I where $\xi = 1$, and $\rho(t) = \rho,\ r(t) = r$ are both constants. For this case, we are able to obtain more explicit results.

Solving (3.3) and (3.4), we get

$$P(t) = \frac{1}{\rho - 2r}[1 - e^{-(\rho-2r)(T-t)}]$$

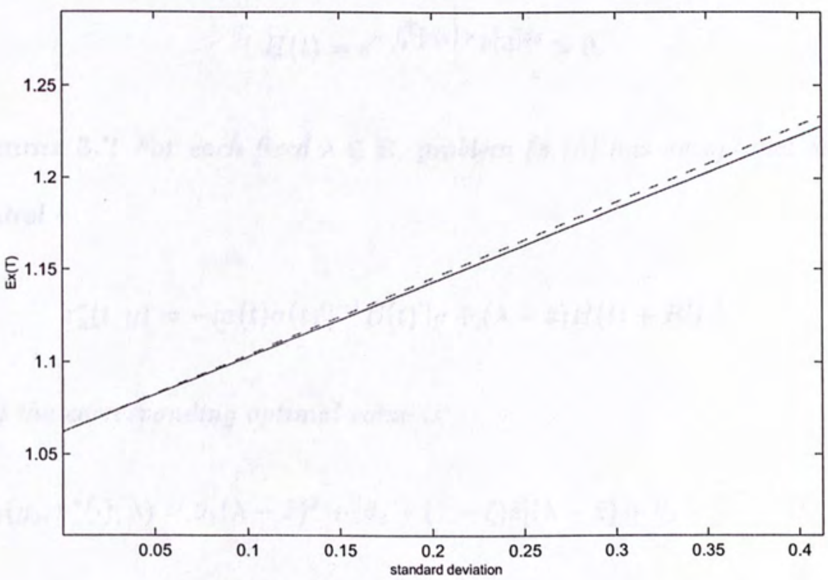
and

$$\begin{aligned} R(t) &= \int_t^T x_0(r-\mu)e^{\mu\tau - \int_t^\tau [r + \frac{\rho-2r}{1-e^{-(\rho-2r)(T-s)}}]ds} d\tau \\ &= x_0(r-\mu)\int_t^T e^{\mu\tau - r(\tau-t) + ln[e^{(\rho-2r)(T-s)}-1]|_t^\tau} d\tau \\ &= \frac{x_0(r-\mu)e^{rt}}{e^{(\rho-2r)(T-t)}-1}\int_t^T e^{(\mu-r)\tau}[e^{(\rho-2r)(T-\tau)} - 1]d\tau \\ &= \frac{x_0e^{rt}}{e^{(\rho-2r)(T-t)}-1}\{\frac{r-\mu}{\mu+r-\rho}[e^{(\mu-r)T} - e^{(\rho-2r)T+(\mu+r-\rho)t}] + [e^{(\mu-r)T} - e^{(\mu-r)t}]\}. \end{aligned}$$

Therefore,

$$\varphi(t) = x_0e^{\mu t} - R(t) = x_0\frac{2r-\rho}{\mu+r-\rho}\frac{e^{rt}[e^{(\mu+r-\rho)T} - e^{(\mu+r-\rho)t}]}{e^{(2r-\rho)T} - e^{(2r-\rho)t}}.$$

Example 3.1 Take the same example as in [24] where a market has a bank account with $r(t) = 6\%$ and only one stock with $b(t) = 12\%$ and $\sigma(t) = 15\%$, using $T = 1$ (*year*) and $x_0 = 1$ (*million*). Through some numerical calculations, we can plot the relationship between terminal return and standard deviation on a two-dimensional diagram, while μ changed from r to $+\infty$. In the following mean-standard-deviation diagram, the spotted straight line is the efficient frontier of the Mean-Variance problem: $Ex(1) = x_0 e^{0.06} + 0.4165\sqrt{\text{Var } x(1)}$ (see [24], formulation (6.11)), and the other line is the tracking frontier $(\sqrt{\text{Var } x(T)},\ Ex(T))$, under the tracking efficient portfolios of Model I.



3.3 Solution to Model II

In this section we turn to Model II. We introduce the following equations:

$$\begin{cases} \dot{P}(t) = [\rho(t) - 2r(t)]P(t) - \xi \\ P(T) = 1 - \xi, \end{cases} \tag{3.19}$$

$$\begin{cases} \dot{H}(t) = [r(t) + \frac{\xi}{P(t)}]H(t) \\ H(T) = 1, \end{cases} \tag{3.20}$$

and

$$\begin{cases} \dot{R}(t) = [r(t) + \frac{\xi}{P(t)}]R(t) - f(t) \\ R(T) = 0. \end{cases} \tag{3.21}$$

Note that, similar to Lemma 3.1, the solutions of (3.19) and (3.20) must satisfy $P(t) > 0,\ \ H(t) > 0,\ \forall\ t \in [0, T]$. Moreover, we can get the solution of (3.20) as

$$H(t) = e^{-\int_t^T [r(s) + \frac{\xi}{P(s)}]ds} > 0. \tag{3.22}$$

Lemma 3.2 *For each fixed $\lambda \in \mathbb{R}$, problem (2.15) has an optimal feedback control*

$$\pi_\lambda^*(t, y) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[y + (\lambda - \hat{z})H(t) + R(t)], \tag{3.23}$$

and the corresponding optimal value is

$$\tilde{J}_2(y_0, \pi_\lambda^*(\cdot); \lambda) = \theta_1(\lambda - \hat{z})^2 + [\theta_2 - (1 - \xi)\hat{z}](\lambda - \hat{z}) + \theta_3 - \frac{1 - \xi}{2}\hat{z}^2, \tag{3.24}$$

where

$$\theta_1 = -\frac{1}{2}\int_0^T \rho(t)P(t)H^2(t)dt < 0, \tag{3.25}$$

$$\theta_2 = \int_0^T P(t)H(t)[f(t) - \rho(t)R(t)]dt, \tag{3.26}$$

$$\theta_3 = \int_0^T P(t)R(t)[f(t) - \frac{1}{2}\rho(t)R(t)]dt. \tag{3.27}$$

Proof. Let $(y(\cdot), \pi(\cdot))$ be any given admissible pair. Applying the Ito formula to $\phi_1(t, y) = \frac{1}{2}P(t)y^2$ and $\phi_2(t, y) = P(t)[(\lambda - \hat{z})H(t) + R(t)]y$, we obtain

$$
\begin{aligned}
&\tfrac{1}{2}d\{P(t)y^2(t)\} \\
= \;& \{\tfrac{1}{2}[(\rho(t)-2r(t))P(t)-\xi]y^2(t)+P(t)y(t)[r(t)y(t)+B(t)\pi(t)+f(t)] \\
& +\tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\}dt+\{\cdots\}dW(t) \\
= \;& \{\tfrac{1}{2}[\rho(t)P(t)-\xi]y^2(t)+P(t)y(t)[B(t)\pi(t)+f(t)] \\
& +\tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\}dt+\{\cdots\}dW(t),
\end{aligned}
\tag{3.28}
$$

and

$$
\begin{aligned}
& d\{P(t)[(\lambda-\hat{z})H(t)+R(t)]y(t)\} \\
= \;& \{[(\rho(t)-2r(t))P(t)-\xi][(\lambda-\hat{z})H(t)+R(t)]y(t) \\
& +[(r(t)P(t)+\xi)[(\lambda-\hat{z})H(t)+R(t)]-f(t)P(t)]y(t) \\
& +P(t)[(\lambda-\hat{z})H(t)+R(t)][r(t)y(t)+B(t)\pi(t)+f(t)]\}dt+\{\cdots\}dW(t) \\
= \;& \{[\rho(t)P(t)[(\lambda-\hat{z})H(t)+R(t)]-f(t)P(t)]y(t) \\
& +P(t)[(\lambda-\hat{z})H(t)+R(t)][B(t)\pi(t)+f(t)]\}dt+\{\cdots\}dW(t).
\end{aligned}
\tag{3.29}
$$

Integrating the above two equations from 0 to T, taking expectations and adding them together, we get

$$
\begin{aligned}
&\tfrac{1}{2}(1-\xi)Ey^2(T)+(1-\xi)(\lambda-\hat{z})Ey(T)\\
= \;& E\int_0^T\{\tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\\
&+P(t)[y(t)+(\lambda-\hat{z})H(t)+R(t)]B(t)\pi(t)\}dt\\
&+E\int_0^T\{\tfrac{1}{2}[\rho(t)P(t)-\xi]y^2(t)+\rho(t)P(t)[(\lambda-\hat{z})H(t)+R(t)]y(t)\\
&+P(t)f(t)[(\lambda-\hat{z})H(t)+R(t)]\}dt\\
= \;& E\int_0^T\{\tfrac{1}{2}P(t)[\pi(t)-\pi_\lambda^*(t,y)]'[\sigma(t)\sigma(t)'][\pi(t)-\pi_\lambda^*(t,y)]\}dt\\
&+E\int_0^T\{P(t)f(t)[(\lambda-\hat{z})H(t)+R(t)]-\tfrac{\xi}{2}y^2(t)\\
&-\tfrac{1}{2}\rho(t)P(t)[(\lambda-\hat{z})H(t)+R(t)]^2\}dt,
\end{aligned}
$$

where $\pi_\lambda^*(t,y)$ is defined as the right-hand side of (3.23). Consequently,

$$
\begin{aligned}
&\tilde{J}_2(y_0,\pi_\lambda(\cdot),\lambda)\\
&= \tfrac{1}{2}\int_0^T P(t)[\pi(t)-\pi_\lambda^*(t,y)]'[\sigma(t)\sigma(t)'][\pi(t)-\pi_\lambda^*(t,y)]dt\\
&\quad+\int_0^T P(t)[(\lambda-\hat{z})H(t)+R(t)]\{f(t)-\tfrac{1}{2}\rho(t)[(\lambda-\hat{z})H(t)+R(t)]\}dt\\
&\quad-\tfrac{1-\xi}{2}[2\hat{z}(\lambda-\hat{z})+\hat{z}^2]\\
&=\tfrac{1}{2}\int_0^T P(t)[\pi(t)-\pi_\lambda^*(t,y)]'[\sigma(t)\sigma(t)'][\pi(t)-\pi_\lambda^*(t,y)]dt\\
&\quad-[\tfrac{1}{2}\int_0^T\rho(t)P(t)H^2(t)dt](\lambda-\hat{z})^2\\
&\quad+[\int_0^T P(t)H(t)[f(t)-\rho(t)R(t)]dt-(1-\xi)\hat{z}](\lambda-\hat{z})\\
&\quad+\int_0^T P(t)R(t)f(t)dt-\tfrac{1}{2}\int_0^T\rho(t)P(t)R^2(t)dt-\tfrac{1-\xi}{2}\hat{z}^2.
\end{aligned}
$$

Since $P(t)>0$, it follows immediately that the optimal feedback control is given by (3.23) and the optimal value is given by (3.24). □

Proposition 3.3 *Problem (2.12) has an optimal feedback control*

$$\pi_2^*(t,y) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[y + (\lambda^* - \hat{z})H(t) + R(t)], \tag{3.30}$$

where

$$\lambda^* - \hat{z} = -\frac{\theta_2 - (1-\xi)\hat{z}}{2\theta_1} = \frac{\int_0^T P(t)H(t)[f(t) - \rho(t)R(t)]dt - (1-\xi)\hat{z}}{\int_0^T \rho(t)P(t)H^2(t)dt}, \tag{3.31}$$

and the corresponding optimal value is

$$J_2(y_0, \pi_2^*(\cdot)) = \theta_3 - \frac{1-\xi}{2}\hat{z}^2 - \frac{[\theta_2 - (1-\xi)\hat{z}]^2}{4\theta_1} \tag{3.32}$$

Proof. Since $J_2(y_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$ and the constraint function $Ey(T) - \hat{z}$ is affine in $\pi(\cdot)$, we can apply the well-known duality theorem (see, e.g., [15]) to conclude that if problem (2.12) is finite for every $\hat{z} \in \mathbb{R}$, then the optimal value is given by (2.14). By Lemma 3.2, $\inf_{\pi(\cdot) \text{ admissible}} \tilde{J}_2(y_0, \pi_\lambda(\cdot); \lambda)$ is a quadratic function in $\lambda - \hat{z}$. Moreover, it is clear that

$$\theta_1 = -\frac{1}{2}\int_0^T \rho(t)P(t)H^2(t)dt < 0.$$

Hence, in view of (2.14), we maximize the quadratic function (3.24) over $\lambda - \hat{z}$ and conclude that the maximizer is given by (3.31), whereas the optimal value and control law are obtained by (3.24) and (3.23) respectively, with $\lambda = \lambda^*$. This leads to (3.32) and (3.30). □

Theorem 3.2 *Problem (2.8) has an optimal feedback control*

$$\pi_2^*(t,x) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[x - e^{\mu t}x_0 + \big(\lambda^* - (z - e^{\mu T}x_0)\big)H(t) + R(t)], \tag{3.33}$$

where

$$\begin{aligned}\lambda^* - (z - e^{\mu T}x_0) &= -\tfrac{\theta_2 - (1-\xi)(z - e^{\mu T}x_0)}{2\theta_1} \\ &= \tfrac{\int_0^T P(t)H(t)[f(t) - \rho(t)R(t)]dt - (1-\xi)(z - e^{\mu T}x_0)}{\int_0^T \rho(t)P(t)H^2(t)dt},\end{aligned} \tag{3.34}$$

and the corresponding optimal value is

$$J_2(x_0, \pi_2^*(\cdot)) = \theta_3 - \frac{1-\xi}{2}(z - e^{\mu T}x_0)^2 - \frac{[\theta_2 - (1-\xi)(z - e^{\mu T}x_0)]^2}{4\theta_1}. \tag{3.35}$$

Proof. Replacing y by $x - x_0 e^{\mu t}$ in Proposition 3.3, we get the desired results. $\square$

Next we investigate the minimum optimal value of J_2 achieved over all the possible $z \in \mathbb{R}$.

Theorem 3.3 *We have*

$$1 - \xi + 2\theta_1 > 0. \tag{3.36}$$

Moreover, the minimum optimal value of (2.8) over $z \in \mathbb{R}$ is

$$J_{2\_min}^* = \theta_3 - \frac{\theta_2^2}{2(1 - \xi + 2\theta_1)}, \tag{3.37}$$

with the corresponding expected terminal wealth

$$z_{2\_min} := \frac{\theta_2}{1 - \xi + 2\theta_1} + e^{\mu T}x_0, \tag{3.38}$$

and the corresponding Lagrange multiplier $\lambda^*_{min} = 0$. Furthermore, the portfolio that achieves the above minimum optimal value is

$$\pi^*_{2\_min}(t,x) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[x - e^{\mu t}x_0 - (z - e^{\mu T}x_0)H(t) + R(t)]. \quad (3.39)$$

Proof. First we prove that $J_2(x_0, \pi^*_2(\cdot))$ as a function of z, denoted by $\psi(z)$, is a strictly convex function in z.

By Theorem 3.2, for any $z_1 \neq z_2$, there exist $\pi^*_{z_1}(\cdot)$ and $\pi^*_{z_2}(\cdot)$ which are different efficient control laws corresponding to z_1 and z_2, i.e.

$$J_2(x_0, \pi^*_{z_1}(\cdot)) = \psi(z_1),$$

$$J_2(x_0, \pi^*_{z_2}(\cdot)) = \psi(z_2).$$

Since $x(\cdot)$ is linear in $\pi(\cdot)$, $\bar{\pi}(\cdot) := \frac{\pi^*_{z_1}(\cdot)+\pi^*_{z_2}(\cdot)}{2}$ must be an admissible control law corresponding to $\bar{z} = \frac{z_1+z_2}{2}$. Noting that $J_2(x_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$, we obtain

$$\psi(\bar{z}) \leq J_2(x_0, \bar{\pi}(\cdot)) < \frac{J_2(x_0, \pi^*_{z_1}(\cdot)) + J_2(x_0, \pi^*_{z_2}(\cdot))}{2} = \frac{\psi(z_1) + \psi(z_2)}{2}.$$

Therefore,

$$\psi(\frac{z_1 + z_2}{2}) < \frac{\psi(z_1) + \psi(z_2)}{2};$$

that is, $\psi(z)$ is strictly convex in z.

We rewrite (3.35) as

$$\begin{aligned} J_2(x_0, \pi_2^*(\cdot)) &= \theta_3 - \tfrac{1-\xi}{2}(z - e^{\mu T}x_0)^2 - \tfrac{[\theta_2-(1-\xi)(z-e^{\mu T}x_0)]^2}{4\theta_1} \\ &= -\tfrac{(1-\xi)(1-\xi+2\theta_1)}{4\theta_1}(z - e^{\mu T}x_0)^2 + \tfrac{\theta_2(1-\xi)}{2\theta_1}(z - e^{\mu T}x_0) + \theta_3 - \tfrac{\theta_2^2}{4\theta_1} \\ &= -\tfrac{(1-\xi)(1-\xi+2\theta_1)}{4\theta_1}[(z - e^{\mu T}x_0) - \tfrac{\theta_2}{1-\xi+2\theta_1}]^2 + \theta_3 - \tfrac{\theta_2^2}{2(1-\xi+2\theta_1)}. \end{aligned}$$

It follows from the proved strict convexity of the optimal value on z, that

$$1 - \xi + 2\theta_1 > 0.$$

Moreover we minimize the quadratic function $J_2(x_0, \pi_2^*(\cdot))$ over z and conclude that the minimizer is given by (3.38), whereas the minimum value and corresponding control law are obtained by (3.37) and (3.39) respectively. □

Proposition 3.4 *(a) If $r(t) \geq \mu, \forall\, t \in [0,T]$, then $\theta_2 \geq 0$. In this case the minimum expected terminal wealth is no less than $e^{\mu T}x_0$.*

(b) If $r(t) < \mu, \forall\, t \in [0,T]$, then $\theta_2 < 0$. In this case the minimum expected terminal wealth is less than $e^{\mu T}x_0$.

Proof. We write the solution of (3.21) as

$$R(t) = -\int_t^T f(\tau) e^{-\int_t^\tau [r(s)+\frac{\xi}{P(s)}]ds} d\tau \tag{3.40}$$

Then, (3.26) becomes

$$\theta_2 = \int_0^T P(t)H(t)\big[f(t) + \rho(t)\int_t^T f(\tau)e^{-\int_t^\tau [r(s)+\frac{\xi}{P(s)}]ds}d\tau\big]dt$$

If $r(t) \geq \mu, \forall\, t \in [0,T]$, then $f(t) = (r(t) - \mu)x_0 e^{\mu t} \geq 0$. Combining (3.36) and (3.38), we get the results of (a) respectively. Similarly we get (b). $\square$

3.4 A Special Case of Model II: Mean-Variance Portfolio Selection

If the investor's objective is to look for an admissible portfolio that minimizes the terminal risk while satisfying the targeted mean payoff at the terminal time, the problem of finding such a portfolio is referred to as the *mean-variance portfolio selection problem.* It turns out that the mean-variance portfolio selection is a special case of Model II.

Definition 3.1 The mean-variance portfolio selection problem is formulated as following:

$$\begin{cases} \text{Minimize } \; J_{MV}(x_0, \pi(\cdot)) = \frac{1}{2}\text{Var } x(T) \\ \text{subject to } \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ admissible,} \end{cases} \end{cases} \tag{3.41}$$

where $z \in \mathbb{R}$ is given.

Moreover, an optimal portfolio to the above problem, if it ever exists, is called an *efficient portfolio* corresponding to z, the corresponding $(\text{Var } x(T), z) \in \mathbb{R}^2$ and $(\sigma_{x(T)}, z) \in \mathbb{R}^2$ are interchangeably called an *efficient point*, where

$\sigma_{x(T)}$ denotes the standard deviation of $x(T)$. The set of all the efficient points is called the *efficient frontier.*

Clearly, this is a special case of problem (2.8) with $\xi = 0$.

Theorem 3.4 *Problem (3.41) has an optimal feedback portfolio*

$$\pi^*(t,x) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[x - \frac{z - x_0 e^{\int_0^T [r(s)-\rho(s)]ds}}{1 - e^{-\int_0^T \rho(s)ds}} e^{-\int_t^T r(s)ds}], \quad (3.42)$$

and the optimal value of Var $x(T)$ *is*

$$\text{Var } x^*(T) = \frac{1}{e^{\int_0^T \rho(s)ds} - 1}[z - x_0 e^{\int_0^T r(s)ds}]^2. \quad (3.43)$$

Proof. By solving (3.19), (3.20) and (3.21) with $\xi = 0$, we get

$P(t) = e^{\int_t^T [2r(s)-\rho(s)]ds}$,

$H(t) = e^{-\int_t^T r(s)ds}$,

$R(t) = e^{\int_0^t r(s)ds} \int_t^T f(\tau) e^{-\int_0^\tau r(s)ds} d\tau = x_0[e^{\mu t} - e^{\mu T - \int_t^T r(s)ds}]$.

Therefore,

$$\begin{aligned}\theta_1 &= -\tfrac{1}{2}\int_0^T \rho(t)P(t)H^2(t)dt \\ &= -\tfrac{1}{2}\int_0^T \rho(t)e^{-\int_t^T \rho(s)ds}dt \\ &= -\tfrac{1}{2}(1 - e^{-\int_0^T \rho(s)ds}),\end{aligned}$$

$$
\begin{aligned}
\theta_2 &= \int_0^T P(t)H(t)[f(t)-\rho(t)R(t)]dt \\
&= x_0\int_0^T e^{\int_t^T[r(s)-\rho(s)]ds}\{(r(t)-\mu)e^{\mu t}-\rho(t)[e^{\mu t}-e^{\mu T-\int_t^T r(s)ds}]\}dt \\
&= x_0\int_0^T\{(r(t)-\rho(t)-\mu)e^{\mu t+\int_t^T[r(s)-\rho(s)]ds}+\rho(t)e^{\mu T-\int_t^T\rho(s)ds}\}dt \\
&= x_0e^{\mu T}\int_0^T\{(r(t)-\rho(t)-\mu)e^{\int_t^T[r(s)-\rho(s)-\mu]ds}+\rho(t)e^{-\int_t^T\rho(s)ds}\}dt \\
&= x_0e^{\mu T}[e^{\int_0^T[r(s)-\rho(s)-\mu]ds}-e^{-\int_0^T\rho(s)ds}] \\
&= x_0e^{-\int_0^T\rho(s)ds}(e^{\int_0^T r(s)ds}-e^{\mu T}),
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_3 &= \int_0^T P(t)R(t)f(t)dt-\tfrac{1}{2}\int_0^T\rho(t)P(t)R^2(t)dt \\
&= \tfrac{1}{2}\int_0^T P(t)R(t)[2f(t)-\rho(t)R(t)]dt \\
&= \tfrac{1}{2}x_0^2\int_0^T e^{\int_t^T[2r(s)-\rho(s)]ds}[e^{\mu t}-e^{\mu T-\int_t^T r(s)ds}] \\
&\quad \{2(r(t)-\mu)e^{\mu t}-\rho(t)[e^{\mu t}-e^{\mu T-\int_t^T r(s)ds}]\}dt \\
&= \tfrac{1}{2}x_0^2e^{2\mu T}\int_0^T\{[2(r(t)-\mu)-\rho(t)]e^{\int_t^T[2r(s)-2\mu-\rho(s)]ds} \\
&\quad -2[r(t)-\mu-\rho(t)]e^{\int_t^T[r(s)-\mu-\rho(s)]ds}-\rho(t)e^{-\int_t^T\rho(s)ds}\}dt \\
&= \tfrac{1}{2}x_0^2e^{2\mu T}\{e^{\int_0^T[2r(s)-2\mu-\rho(s)]ds}-2e^{\int_0^T[r(s)-\mu-\rho(s)]ds}+e^{-\int_0^T\rho(s)ds}\}.
\end{aligned}
$$

Thus, (3.34) becomes

$$\begin{aligned}
&\lambda^* - (z - e^{\mu T}x_0) \\
&= -\frac{\theta_2 - (z - e^{\mu T}x_0)}{2\theta_1} \\
&= \frac{x_0 e^{-\int_0^T \rho(s)ds}(e^{\int_0^T r(s)ds} - e^{\mu T}) - (z - e^{\mu T}x_0)}{1 - e^{-\int_0^T \rho(s)ds}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& -e^{\mu t}x_0 + (\lambda^* - (z - e^{\mu T}x_0))H(t) + R(t) \\
= & -x_0 e^{\mu T - \int_t^T r(s)ds} + \frac{x_0 e^{-\int_0^T \rho(s)ds}(e^{\int_0^T r(s)ds} - e^{\mu T}) - (z - e^{\mu T}x_0)}{1 - e^{-\int_0^T \rho(s)ds}} e^{-\int_t^T r(s)ds} \\
= & -\frac{z - x_0 e^{\int_0^T [r(s) - \rho(s)]ds}}{1 - e^{-\int_0^T \rho(s)ds}} e^{-\int_t^T r(s)ds}.
\end{aligned}$$

So (3.42) follows from (3.33). Finally, (3.43) is obtained from (3.35) by calculating

$$x_0 e^{\mu T} + \frac{\theta_2}{1 - \xi + 2\theta_1} = x_0 e^{\mu T} + x_0(e^{\int_0^T r(s)ds} - e^{\mu T}) = x_0 e^{\int_0^T r(s)ds},$$

and

$$\begin{aligned}
\theta_3 - \frac{\theta_2^2}{2(1 - \xi + 2\theta_1)} = & \ \tfrac{1}{2}x_0^2 e^{-\int_0^T \rho(s)ds}(e^{\int_0^T r(s)ds} - e^{\mu T})^2 \\
& + \tfrac{1}{2}x_0^2 e^{2\mu T}\{e^{\int_0^T [2r(s) - 2\mu - \rho(s)]ds} - 2e^{\int_0^T [r(s) - \mu - \rho(s)]ds} \\
& + e^{-\int_0^T \rho(s)ds}\} \\
= & \ 0.
\end{aligned}$$

The proof is complete. □

Remark 3.1 Theorem 3.4 recovers the results of [24] which uses an embedding technique.

3.5 Solution to Model III

We introduce the following equations:

$$\begin{cases} \dot{P}(t) = [\rho(t) - 2r(t)]P(t) - 1 \\ P(T) = 0, \end{cases} \tag{3.44}$$

$$\begin{cases} \dot{g}_1(t) = [\rho(t) - r(t)]g_1(t) \\ g_1(T) = 1, \end{cases} \tag{3.45}$$

and

$$\begin{cases} \dot{g}_2(t) = [\rho(t) - r(t)]g_2(t) - f(t)P(t) \\ g_2(T) = 0. \end{cases} \tag{3.46}$$

Note that the solutions of (3.44) and (3.45) must satisfy $P(t) > 0, \forall\, t \in [0, T)$, and $g_1(t) > 0,\ \forall\, t \in [0, T]$.

Lemma 3.3 *For each fixed $\hat{\lambda} \in \mathbb{R}$, problem (2.16) has an optimal feedback control*

$$\pi^*_{\hat{\lambda}}(t, y) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[y + \frac{\hat{\lambda}g_1(t) + g_2(t)}{P(t)}], \tag{3.47}$$

and the corresponding optimal value is

$$\tilde{J}_3(y_0, \pi^*_{\hat{\lambda}}(\cdot)) = \theta_1'\hat{\lambda}^2 + [\theta_2' - \hat{z}]\hat{\lambda} + \theta_3', \tag{3.48}$$

where

$$\theta_1' = -\int_0^T \frac{\rho(t)g_1^2(t)}{2P(t)}dt < 0, \tag{3.49}$$

$$\theta_2' = \int_0^T g_1(t)[f(t) - \frac{\rho(t)g_2(t)}{P(t)}]dt, \tag{3.50}$$

$$\theta_3' = \int_0^T g_2(t)[f(t) - \frac{\rho(t)g_2(t)}{2P(t)}]dt. \tag{3.51}$$

Proof. Let $(y(\cdot), \pi(\cdot))$ be any given admissible pair. Applying the Ito formula to $\phi_1(t,y) = \frac{1}{2}P(t)y^2$ and $\phi_2'(t,y) = [\hat{\lambda}g_1(t) + g_2(t)]y$, we obtain

$$\begin{aligned}
&\tfrac{1}{2}d\{P(t)y^2(t)\} \\
= \;& \{\tfrac{1}{2}[\rho(t)P(t) - 1]y^2(t) + P(t)y(t)[B(t)\pi(t) + f(t)] \\
&+ \tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t)\}dt + \{\cdots\}dW(t),
\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}
&d\{[\hat{\lambda}g_1(t) + g_2(t)]y(t)\} \\
= \;& \{[[\rho(t) - r(t)][\hat{\lambda}g_1(t) + g_2(t)] - f(t)P(t)]y(t) \\
&+ [\hat{\lambda}g_1(t) + g_2(t)][r(t)y(t) + B(t)\pi(t) + f(t)]\}dt + \{\cdots\}dW(t) \\
= \;& \{[\rho(t)[\hat{\lambda}g_1(t) + g_2(t)] - f(t)P(t)]y(t) \\
&+ [\hat{\lambda}g_1(t) + g_2(t)][B(t)\pi(t) + f(t)]\}dt + \{\cdots\}dW(t).
\end{aligned} \tag{3.53}$$

Integrating the above two equations from 0 to T, taking expectations and adding them together, we get

$$\begin{aligned}
&\hat{\lambda}Ey(T) \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t)\pi(t)'[\sigma(t)\sigma(t)']\pi(t) + [y(t) + \hat{\lambda}g_1(t) + g_2(t)]B(t)\pi(t)\}dt \\
&+ E\int_0^T \{\tfrac{1}{2}[\rho(t)P(t) - 1]y^2(t) + [\hat{\lambda}g_1(t) + g_2(t)][\rho(t)y(t) + f(t)]\}dt \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t)[\pi(t) - \pi^*_{\hat{\lambda}}(t,y)]'[\sigma(t)\sigma(t)'][\pi(t) - \pi^*_{\hat{\lambda}}(t,y)]\}dt \\
&+ E\int_0^T \{f(t)[\hat{\lambda}g_1(t) + g_2(t)] - \tfrac{1}{2}y^2(t) - \tfrac{\rho(t)[\hat{\lambda}g_1(t) + g_2(t)]^2}{2P(t)}\}dt,
\end{aligned}$$

where $\pi^*_{\hat{\lambda}}(t,y)$ is defined on the right-hand side of (3.47). Hence,

$$\begin{aligned}
&\tilde{J}_3(y_0, \pi_{\hat{\lambda}}(\cdot), \hat{\lambda}) \\
= \ & E\int_0^T \{\tfrac{1}{2}P(t)[\pi(t) - \pi^*_{\hat{\lambda}}(t,y)]'[\sigma(t)\sigma(t)'][\pi(t) - \pi^*_{\hat{\lambda}}(t,y)]\}dt \\
&+ E\int_0^T \{f(t)[\hat{\lambda}g_1(t) + g_2(t)] - \tfrac{\rho(t)[\hat{\lambda}g_1(t)+g_2(t)]^2}{2P(t)}\}dt - \hat{\lambda}\hat{z}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\tilde{J}_3(y_0, \pi^*_{\hat{\lambda}}(\cdot), \hat{\lambda}) \\
= \ & E\int_0^T \{f(t)[\hat{\lambda}g_1(t) + g_2(t)] - \tfrac{\rho(t)[\hat{\lambda}g_1(t)+g_2(t)]^2}{2P(t)}\}dt - \hat{\lambda}\hat{z} \\
= \ & [-\int_0^T \tfrac{\rho(t)g_1^2(t)}{2P(t}) dt]\hat{\lambda}^2 + \{\int_0^T [g_1(t)f(t) - \tfrac{\rho(t)g_1(t)g_2(t)}{P(t)}]dt - \hat{z}\}\hat{\lambda} \\
&+ \int_0^T g_2(t)f(t)dt - \int_0^T \tfrac{\rho(t)g_2^2(t)}{2P(t)}dt.
\end{aligned}$$

Using the similar argument as that in the proof of Lemma 3.2, we complete the proof. □

Theorem 3.5 *Problem (2.9) has an optimal feedback control*

$$\pi^*_3(t,x) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[x - e^{\mu t}x_0 + \frac{\hat{\lambda}^* g_1(t) + g_2(t)}{P(t)}], \tag{3.54}$$

where

$$\hat{\lambda}^* = -\frac{\theta_2' - (z - e^{\mu T}x_0)}{2\theta_1'} = \frac{\int_0^T g_1(t)[f(t) - \frac{\rho(t)g_2(t)}{P(t)}]dt - (z - e^{\mu T}x_0)}{\int_0^T \frac{\rho(t)g_1^2(t)}{P(t)}dt}, \tag{3.55}$$

and the corresponding optimal value is

$$J_3(x_0, \pi^*_3(\cdot)) = \theta_3' - \frac{[\theta_2' - (z - e^{\mu T}x_0)]^2}{4\theta_1'}. \tag{3.56}$$

Proof. We can use the similar argument in the proof of Proposition 3.3. Since $\theta_1' < 0$, we maximize the quadratic function (3.48) over $\hat{\lambda} - \hat{z}$ and conclude that the maximizer is given by

$$\hat{\lambda}^* = -\frac{\theta_2' - \hat{z}}{2\theta_1'} = \frac{\int_0^T g_1(t)[f(t) - \frac{\rho(t)g_2(t)}{P(t)}]dt - \hat{z}}{\int_0^T \frac{\rho(t)g_1^2(t)}{P(t)}dt},$$

whereas the optimal value and control law are obtained by

$$J_3(y_0, \pi_3^*(\cdot)) = \theta_3' - \frac{[\theta_2' - \hat{z}]^2}{4\theta_1'},$$

and

$$\pi_3^*(t, y) = -[\sigma(t)\sigma(t)']^{-1}B(t)'[y + \frac{\hat{\lambda}^* g_1(t) + g_2(t)}{P(t)}].$$

Replacing $y = x - x_0e^{\mu t}$ and $\hat{z} = z - e^{\mu T}x_0$ in the above results, we get (3.56) and (3.54) respectively, with (3.55). □

Remark 3.2 Comparing Theorem 3.5 to Theorem 3.2, we see that the optimal solution of Model III is not a limit of that of Model II as $\xi \to 1$.

The expression (3.56) also discloses the minimum optimal value of J_3 achieved over all the possible $z \in \mathbb{R}$.

Theorem 3.6 *The minimum optimal value of (2.9) is*

$$J_{3\_min}^* = \theta_3', \tag{3.57}$$

with the corresponding expected terminal wealth

$$z_{3\_min} := \theta_2' + e^{\mu T} x_0, \tag{3.58}$$

and the corresponding Lagrange multiplier $\hat{\lambda}^*_{min} = 0$*. Moreover, the portfolio that achieves the above minimum optimal value is*

$$\pi^*_{3\_min}(t,x) = -[\sigma(t)\sigma(t)']^{-1} B(t)'[x - e^{\mu t} x_0 + \frac{g_2(t)}{P(t)}]. \tag{3.59}$$

Proof. Since (3.56) is a quadratic function in z, and the coefficient of second order term is greater than zero ($\theta_1' < 0$), we can get (3.57) and (3.58) immediately. The assertion $\hat{\lambda}^*_{min} = 0$ can be verified via (3.55). Moreover, (3.59) follows from (3.54). □

Chapter 4

Efficient Tracking: Markov-Modulated Market Parameters

In this chapter, we investigate the case where there is *regime switching* in the market. More specifically, the interest rate and the appreciation and volatility rates of the stocks randomly depend on the market mode that switches among a finite number of states. Here, the random regime switching is assumed to be independent of the random sources that drive the stock prices. This essentially renders the underlying market *incomplete*. A Markov-chain modulated diffusion formulation is employed to model the problem.

4.1 Problem Formulation

In addition to the complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ along with the Brownian motion $W(t)$, there is a continuous-time stationary Markov chain $\alpha(t)$ taking value in a finite state space $M = \{1, 2, \cdots, l\}$. Moreover, $\alpha(t)$ and $W(t)$ are independent of each other. The Markov chain has a generator $Q = (q_{ij})_{l\times l}$ and stationary transition probabilities

$$p_{ij}(t) = P(\alpha(t) = j|\alpha(0) = i), \ t \geq 0, \ i, j = 1, 2, \cdots, l. \tag{4.1}$$

The filtration $\mathcal{F}_t = \sigma\{W(s), \alpha(s) : 0 \leq s \leq t\}$.

The market parameters in (2.1) and (2.2) take the form

$$r(t) = r(t, \alpha(t)), \ b_m(t) = b_m(t, \alpha(t)), \ \sigma_{mn}(t) = \sigma_{mn}(t, \alpha(t)), \tag{4.2}$$

where $r(t,i)$, $b_m(t,i)$, $\sigma_m(t,i) := (\sigma_{m1}(t,i), \cdots, \sigma_{md}(t,i))$, are given market data corresponding to the market mode $\alpha(t) = i \in \{1, 2, \cdots, l\}$. We rewrite (2.1) and (2.2) respectively as

$$\begin{cases} dp_0(t) = r(t, \alpha(t))p_0(t)dt, \ t \in [0,T] \\ p_0(0) = p_0 > 0, \end{cases} \tag{4.3}$$

and

$$\begin{cases} dp_m(t) = p_m(t)\{b_m(t,\alpha(t))dt + \sum_{n=1}^{d} \sigma_{mn}(t,\alpha(t))dW^n(t)\}, \ t \in [0,T] \\ p_m(0) = p_m > 0, \ m = 1, 2, \cdots, d. \end{cases} \tag{4.4}$$

Define the volatility matrix

$$\sigma(t,i) := (\sigma_{mn}(t,i))_{d\times d}, \tag{4.5}$$

and we assume the following non-degeneracy condition

$$\sigma(t,i)\sigma(t,i)' \geq \delta I, \ \forall t \in [0,T], \ i = 1, 2, \cdots, l \tag{4.6}$$

for some $\delta > 0$. We also assume that all the functions $r(t,i)$, $b_m(t,i)$, $\sigma_{mn}(t,i)$ are measurable and uniformly bounded in t.

Suppose the initial market mode $\alpha(0) = i_0$. Consider an agent with an initial wealth $x_0 > 0$. These initial conditions are fixed throughout this chapter.

Setting

$$B(t,i) := (b_1(t,i) - r(t,i), \cdots, b_d(t,i) - r(t,i)), \quad i = 1, 2, \cdots, l, \tag{4.7}$$

then the wealth equation (2.4) can be written as

$$\begin{cases} dx(t) = [r(t,\alpha(t))x(t) + B(t,\alpha(t))\pi(t)]dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\ x(0) = x_0,\ \alpha(0) = i_0. \end{cases} \tag{4.8}$$

Here we reformulate Models I-III for the market with regime switching.

Definition 4.1 Model I refers to the following optimization problem:

$$\begin{cases} \text{Minimize } \ J_1(x_0, i_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T [x(t) - e^{\mu t}x_0]^2 dt + \frac{1-\xi}{2} E[x(T) - e^{\mu T}x_0]^2 \\ \text{subject to } \ (x(\cdot), \pi(\cdot)) \text{ admissible and satisfying (4.8),} \end{cases} \tag{4.9}$$

where $\mu > 0$ is a given growth rate and $0 \leq \xi \leq 1$.

Definition 4.2 Model II refers to the following optimization problem:

$$\begin{cases} \text{Minimize } \ J_2(x_0, i_0, \pi(\cdot)) = \frac{\xi}{2} E \int_0^T [x(t) - e^{\mu t}x_0]^2 dt + \frac{1-\xi}{2} \text{ Var } x(T) \\ \text{subject to } \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ admissible and satisfying (4.8),} \end{cases} \end{cases} \tag{4.10}$$

where $\mu > 0$ is a deterministic growth rate, $0 \leq \xi < 1$, and $z \in \mathbb{R}$ is given.

Definition 4.3 Model III refers to the following optimization problem:

$$\begin{cases} \text{Minimize } \ J_3(x_0, i_0, \pi(\cdot)) = \frac{1}{2} E \int_0^T [x(t) - e^{\mu t}x_0]^2 dt \\ \text{subject to } \begin{cases} Ex(T) = z \\ (x(\cdot), \pi(\cdot)) \text{ admissible and satisfying (4.8),} \end{cases} \end{cases} \tag{4.11}$$

where $\mu > 0$ is a deterministic growth rate and $z \in \mathbb{R}$ is given.

If we use the same revised method as in Section 2.2, (2.10) is now in the form

$$\begin{cases} dy(t) = [r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t) + f(t,\alpha(t))]dt + \pi'(t)\sigma(t,\alpha(t))dW(t) \\ y(0) = y_0 = 0, \quad \alpha(0) = i_0 \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d), \end{cases} \tag{4.12}$$

where $f(t,i) = (r(t,i) - \mu)x_0 e^{\mu t}$.

Then, Model I is equivalent to

$$\begin{cases} \text{Minimize } \ J_1(y_0,i_0,\pi(\cdot)) = \frac{\xi}{2}E\int_0^T y^2(t)dt + \frac{1-\xi}{2}E[y^2(T)], \quad \xi \in [0,1] \\ \text{subject to } \ (y(\cdot),\pi(\cdot)) \text{ admissible and satisfying (4.12)}, \end{cases} \tag{4.13}$$

Model II is equivalent to

$$\begin{cases} \text{Minimize } \ J_2(y_0,i_0,\pi(\cdot)) = \frac{\xi}{2}E\int_0^T y^2(t)dt + \frac{1-\xi}{2}E[y(T) - \hat{z}]^2, \quad \xi \in [0,1) \\ \text{subject to } \begin{cases} Ey(T) = \hat{z} \\ (y(\cdot),\pi(\cdot)) \text{ admissible and satisfying (4.12)}, \end{cases} \end{cases} \tag{4.14}$$

and Model III is equivalent to

$$\begin{cases} \text{Minimize } \ J_3(y_0,i_0,\pi(\cdot)) = \frac{1}{2}E\int_0^T y^2(t)dt \\ \text{subject to } \begin{cases} Ey(T) = \hat{z} \\ (y(\cdot),\pi(\cdot)) \text{ admissible and satisfying (4.12)}. \end{cases} \end{cases} \tag{4.15}$$

Moreover, (2.15) and (2.16) can be rewritten as

$$\begin{cases} \text{Minimize } \ \tilde{J}_2(y_0,i_0,\pi(\cdot);\lambda) = \frac{\xi}{2}E\int_0^T y^2(t)dt + \frac{1-\xi}{2}\{E[y(T)+\lambda-\hat{z}]^2 - \lambda^2\} \\ \text{subject to } \ (y(\cdot),\pi(\cdot)) \text{ admissible and satisfying (4.12)}, \end{cases} \tag{4.16}$$

and

$$\begin{cases} \text{Minimize } \ \tilde{J}_3(y_0,i_0,\pi(\cdot);\hat{\lambda}) = \frac{1}{2}E\int_0^T y^2(t)dt + \hat{\lambda}E(y(T)-\hat{z}) \\ \text{subject to } \ (y(\cdot),\pi(\cdot)) \text{ admissible and satisfying (4.12)}. \end{cases} \tag{4.17}$$

Before we proceed, we introduce the following generalized Ito lemma (see [2]) for Markov-modulated processes.

Lemma 4.1 *Given an n-dimensional process $x(\cdot)$ satisfying*

$$dx(t) = b(t, x(t), \alpha(t))dt + \sigma(t, x(t), \alpha(t))dW(t),$$

and a number of function $\varphi(\cdot, \cdot, i) \in C^2([0, T] \times \mathbb{R}^n)$, $i = 1, 2, \cdots, l$, we have

$$d\varphi(t, x(t), \alpha(t)) = \Gamma_\varphi(t, x(t), \alpha(t))dt + \varphi_x(t, x(t), \alpha(t))'\sigma(t, x(t), \alpha(t))dW(t),$$

where

$$\begin{aligned}\Gamma_\varphi(t, x, i) &= \varphi_t(t, x, i) + \varphi_x(t, x, i)'b(t, x, i) \\ &\quad + \tfrac{1}{2}tr[\sigma(t, x, i)'\varphi_{xx}(t, x, i)\sigma(t, x, i)] + \textstyle\sum_{j=1}^{l} q_{ij}\varphi(t, x, j).\end{aligned}$$

Denote

$$\rho(t, i) := B(t, i)[\sigma(t, i)\sigma(t, i)']^{-1}B(t, i)', \quad i = 1, 2, \cdots, l. \tag{4.18}$$

Proposition 4.1 *([25], Theorem 3.1) The problem (2.8) (or (2.9)) is feasible for every $z \in \mathbb{R}$ if and only if*

$$E\int_0^T \mid B(t, \alpha(t)) \mid^2 dt > 0. \tag{4.19}$$

4.2 Solution to Model I with Regime Switching

We introduce the following two systems of ODEs:

$$\begin{cases} \dot{P}(t,i) = & [\rho(t,i) - 2r(t,i)]P(t,i) - \sum_{j=1}^{l} q_{ij}P(t,j) - \xi \\ P(T,i) = & 1-\xi, \quad i = 1,2,\cdots,l, \end{cases} \tag{4.20}$$

and

$$\begin{cases} \dot{R}(t,i) = & [r(t,i) + \frac{\xi}{P(t,i)}]R(t,i) - f(t,i) \\ & -\frac{1}{P(t,i)}\sum_{j=1}^{l} q_{ij}P(t,j)[R(t,j) - R(t,i)] \\ R(T,i) = & 0, \quad i = 1,2,\cdots,l. \end{cases} \tag{4.21}$$

Proposition 4.2 *The solutions of (4.20) must satisfy*

(*a*) *If* $0 \le \xi < 1$, *then* $P(t,i) > 0$, $\forall\, t \in [0,T]$, $i = 1,2,\cdots,l$, *and*

(*b*) *If* $\xi = 1$, *then* $P(t,i) > 0$, $\forall\, t \in [0,T)$, $i = 1,2,\cdots,l$.

Proof. Note that equation (4.20) can be written as

$$\begin{cases} \dot{P}(t,i) = [\rho(t,i) - 2r(t,i) - q_{ii}]P(t,i) - \sum_{j\neq i}^{l} q_{ij}P(t,j) - \xi \\ P(T,i) = 1-\xi, \quad i = 1,2,\cdots,l. \end{cases} \tag{4.22}$$

Treating this as a system of terminal-valued ODEs, a variation-of-constant formula yields

$$\begin{aligned} P(t,i) = & \ (1-\xi)e^{-\int_t^T[\rho(s,i)-2r(s,i)-q_{ii}]ds} \\ & + \int_t^T e^{-\int_t^\tau[\rho(s,i)-2r(s,i)-q_{ii}]ds}[\textstyle\sum_{j\neq i}^{l} q_{ij}P(\tau,j) + \xi]d\tau, \\ & \qquad i = 1,2,\cdots,l. \end{aligned} \tag{4.23}$$

(*a*) If $0 \le \xi < 1$, construct a sequence $\{P^{(k)}(\cdot,i)\}$ (known as the Picard sequence) as follows

$$\begin{aligned} & P^{(0)}(t,i) = 1-\xi, \ t \in [0,T], \ i = 1,2,\cdots,l, \\ & P^{(k+1)}(t,i) = \ (1-\xi)e^{-\int_t^T[\rho(s,i)-2r(s,i)-q_{ii}]ds} \\ & \qquad + \int_t^T e^{-\int_t^\tau[\rho(s,i)-2r(s,i)-q_{ii}]ds}[\textstyle\sum_{j\neq i}^{l} q_{ij}P^{(k)}(\tau,j) + \xi]d\tau, \\ & \qquad t \in [0,T], \ i = 1,2,\cdots,l, \ k = 0,1,\cdots. \end{aligned}$$

Noting $q_{ij} \geq 0$ for all $j \neq i$, we have

$P^{(k)}(t,i) \geq (1-\xi)e^{-\int_t^T[\rho(s,i)-2r(s,i)-q_{ii}]ds} > 0, \quad k = 0, 1, \cdots.$

On the other hand, it is well-known that $P(t,i)$ is the limit of the Picard sequence $\{P^{(k)}(t,i)\}$ as $k \longrightarrow \infty$. Thus $P(t,i) > 0, \ \forall\, t \in [0,T]$.

(*b*) If $\xi = 1$, $\{P^{(k)}(\cdot,i)\}$ is constructed as

$$
\begin{aligned}
P^{(0)}(t,i) &= 0, \ t \in [0,T), \ i = 1, 2, \cdots, l, \\
P^{(k+1)}(t,i) &= \int_t^T e^{-\int_t^\tau[\rho(s,i)-2r(s,i)-q_{ii}]ds}[\textstyle\sum_{j\neq i}^l q_{ij}P^{(k)}(\tau,j)+1]d\tau, \\
&\quad t \in [0,T), \ i = 1, 2, \cdots, l, \ k = 0, 1, \cdots.
\end{aligned}
$$

Similarly, we have

$P^{(k)}(t,i) \geq \int_t^T e^{-\int_t^\tau[\rho(s,i)-2r(s,i)-q_{ii}]ds}d\tau > 0, \quad k = 0, 1, \cdots.$

Therefore, $P(t,i) > 0, \ \forall\, t \in [0,T)$, with only $P(T,i) = 0$. □

Proposition 4.3 *Problem (4.13) has an optimal feedback control*

$$\pi_1^*(t,y,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[y+R(t,i)], \tag{4.24}$$

and the corresponding optimal value is

$$J_1(y_0,i_0,\pi_1^*(\cdot)) = \int_0^T \sum_{i=1}^l p_{i_0 i}(t)P(t,i)R(t,i)[f(t,i) - \frac{1}{2}\rho(t,i)R(t,i)]dt. \tag{4.25}$$

Proof. Let $(y(\cdot),\pi(\cdot))$ is any given admissible pair. Applying the Ito formula (Lemma 4.1) to $\phi_3(t,y,i) = \frac{1}{2}P(t,i)y^2$ and $\phi_4(t,y,i) = yP(t,i)R(t,i)$, we obtain

$$
\begin{aligned}
&\tfrac{1}{2}d\{P(t,\alpha(t))y^2(t)\} \\
= &\ \tfrac{1}{2}\{P_t(t,\alpha(t))y^2(t)dt + 2P(t,\alpha(t))y(t)dy \\
&+P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t)dt \\
&+\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)y^2(t)dt\} + \{\cdots\}dW(t) \\
= &\ \tfrac{1}{2}\{\big[[\rho(t,\alpha(t)) - 2r(t,\alpha(t))]P(t,\alpha(t)) - \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j) - \xi\big]y^2(t) \\
&+2P(t,\alpha(t))y(t)[r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t) + f(t,\alpha(t))] \\
&+P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t) + \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)y^2(t)\}dt \\
&+\{\cdots\}dW(t) \\
= &\ \tfrac{1}{2}\{[\rho(t,\alpha(t))P(t,\alpha(t)) - \xi]y^2(t) \\
&+2P(t,\alpha(t))y(t)[B(t,\alpha(t))\pi(t) + f(t,\alpha(t))] \\
&+P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t)\}dt + \{\cdots\}dW(t),
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
& d\{y(t)P(t,\alpha(t))R(t,\alpha(t))\} \\
= \;& P(t,\alpha(t))R(t,\alpha(t))dy + y(t)\{P_t(t,\alpha(t))R(t,\alpha(t)) + P(t,\alpha(t))R_t(t,\alpha(t))\}dt \\
& + \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}y(t)P(t,j)R(t,j)dt + \{\cdots\}dW(t) \\
= \;& P(t,\alpha(t))R(t,\alpha(t))[r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t) + f(t,\alpha(t))]dt \\
& + y(t)\{R(t,\alpha(t))\,[[\rho(t,\alpha(t)) - 2r(t,\alpha(t))]P(t,\alpha(t)) \\
& - \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j) - \xi] + [r(t,\alpha(t))P(t,\alpha(t)) + \xi]R(t,\alpha(t)) \\
& - \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[R(t,j) - R(t,\alpha(t))] - f(t,\alpha(t))P(t,\alpha(t))\}dt \\
& + \textstyle\sum_{j=1}^{l} q_{\alpha(t)j}y(t)P(t,j)R(t,j)dt + \{\cdots\}dW(t) \\
= \;& \{y(t)P(t,\alpha(t))[\rho(t,\alpha(t))R(t,\alpha(t)) - f(t,\alpha(t))] \\
& + [B(t,\alpha(t))\pi(t) + f(t,\alpha(t))]P(t,\alpha(t))R(t,\alpha(t))\}dt + \{\cdots\}dW(t).
\end{aligned}
\tag{4.27}
$$

Integrating both (4.26) and (4.27) from 0 to T, taking expectations, adding them together, we obtain

$$
\begin{aligned}
& \tfrac{1}{2}(1-\xi)Ey^2(T) \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t,\alpha(t))[\pi(t) - \pi_1^*(t,y(t),\alpha(t))]'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))'] \\
& [\pi(t) - \pi_1^*(t,y(t),\alpha(t))] - \tfrac{\xi}{2}y^2(t) \\
& + P(t,\alpha(t))R(t,\alpha(t))[f(t,\alpha(t)) - \tfrac{1}{2}\rho(t,\alpha(t))R(t,\alpha(t))]\}dt
\end{aligned}
$$

where $\pi_1^*(t, y, i)$ is defined as the right-hand side of (4.24). Then

$$
\begin{aligned}
& J_1(y_0, i_0, \pi(\cdot), \lambda) \\
= \; & E\int_0^T \{\tfrac{1}{2}P(t, \alpha(t))[\pi(t) - \pi_1^*(t, y(t), \alpha(t))]' \\
& [\sigma(t, \alpha(t))\sigma(t, \alpha(t))'][\pi(t) - \pi_1^*(t, y(t), \alpha(t))] \\
& +P(t, \alpha(t))R(t, \alpha(t))[f(t, \alpha(t)) - \tfrac{1}{2}\rho(t, \alpha(t))R(t, \alpha(t))]\}dt \\
= \; & \int_0^T \sum_{i=1}^l p_{i_0 i}(t)\{\tfrac{1}{2}P(t, i)[\pi(t) - \pi_1^*(t, y, i)]'[\sigma(t, i)\sigma(t, i)'] \\
& [\pi(t) - \pi_1^*(t, y, i)] + P(t, i)R(t, i)[f(t, i) - \tfrac{1}{2}\rho(t, i)R(t, i)]\}dt.
\end{aligned}
$$

Since $P(t, i) > 0$ by Proposition 4.2, it follows immediately that the optimal feedback control is given by (4.24) and the optimal value is given by (4.25), provided that the corresponding equation (4.12) under the feedback control (4.24) has a solution. However, under (4.24), the system (4.12) is a nonhomogeneous linear stochastic differential equation with coefficients modulated by $\alpha(t)$. Since all the coefficients of this linear equation are uniformly bounded and $\alpha(t)$ is independent of $W(t)$, the existence and uniqueness of the solution to the equation are straightforward based on a standard successive approximation scheme. □

Theorem 4.1 *Problem (4.9) has an optimal feedback control*

$$\pi_1^*(t, x, i) = -[\sigma(t, i)\sigma(t, i)']^{-1}B(t, i)'[x - x_0e^{\mu t} + R(t, i)], \tag{4.28}$$

and the corresponding optimal value is

$$J_1(x_0, i_0, \pi_1^*(\cdot)) = \int_0^T \sum_{i=1}^{l} p_{i_0 i}(t) P(t,i) R(t,i)[f(t,i) - \frac{1}{2}\rho(t,i)R(t,i)]dt. \tag{4.29}$$

Proof. Replacing $y = x - x_0 e^{\mu t}$ in Proposition 4.3, we get the result immediately. □

4.3 Solution to Model II with Regime Switching

In this section we turn to Model II for the market with regime switching.

Consider the following three systems of ODEs:

$$\begin{cases} \dot{P}(t,i) = [\rho(t,i) - 2r(t,i)]P(t,i) - \sum_{j=1}^{l} q_{ij} P(t,j) - \xi \\ P(T,i) = 1 - \xi, \quad i = 1, 2, \cdots, l, \end{cases} \tag{4.30}$$

$$\begin{cases} \dot{H}(t,i) = & [r(t,i) + \frac{\xi}{P(t,i)}]H(t,i) \\ & -\frac{1}{P(t,i)} \sum_{j=1}^{l} q_{ij} P(t,j)[H(t,j) - H(t,i)] \\ H(T,i) = & 1, \quad i = 1, 2, \cdots, l, \end{cases} \tag{4.31}$$

and

$$\begin{cases} \dot{R}(t,i) = & [r(t,i) + \frac{\xi}{P(t,i)}]R(t,i) - f(t,i) \\ & -\frac{1}{P(t,i)} \sum_{j=1}^{l} q_{ij} P(t,j)[R(t,j) - R(t,i)] \\ R(T,i) = & 0, \quad i = 1, 2, \cdots, l. \end{cases} \tag{4.32}$$

Proposition 4.4 *The solutions of (4.30) and (4.31) must satisfy* $P(t,i) > 0$, $H(t,i) > 0$, $\forall\, t \in [0,T]$, $i = 1, 2, \cdots, l$.

Proof. Since $0 \leq \xi < 1$, that $P(t,i) > 0$ follows from (a) of Proposition 4.2. By using the same argument, we can show $H(t,i) > 0$, noting that $\frac{P(t,j)}{P(t,i)} > 0$.

□

Lemma 4.2 *For each fixed* $\lambda \in \mathbb{R}$, *problem (4.16) has an optimal feedback control*

$$\pi^*_\lambda(t,y,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[y + (\lambda - \hat{z})H(t,i) + R(t,i)], \quad (4.33)$$

and the corresponding optimal value is

$$\tilde{J}_2(y_0, i_0, \pi^*_\lambda(\cdot)) = \theta_4(\lambda - \hat{z})^2 + [\theta_5 - (1-\xi)\hat{z}](\lambda - \hat{z}) + \theta_6 - \frac{1-\xi}{2}\hat{z}^2, \quad (4.34)$$

where

$$\theta_4 = -\frac{1}{2}\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)\rho(t,i)P(t,i)H^2(t,i)dt < 0, \quad (4.35)$$

$$\theta_5 = \int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)P(t,i)H(t,i)[f(t,i) - \rho(t,i)R(t,i)]dt, \quad (4.36)$$

$$\theta_6 = \int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)P(t,i)R(t,i)[f(t,i) - \frac{1}{2}\rho(t,i)R(t,i)]dt. \quad (4.37)$$

Proof. Let $(y(\cdot), \pi(\cdot))$ is any given admissible pair. Applying the Ito formula to $\phi_3(t,y,i) = \frac{1}{2}P(t,i)y^2$ and $\phi_5(t,y,i) = yP(t,i)[(\lambda - \hat{z})H(t,i) + R(t,i)]$, we obtain (4.26) as well as the following equality

$$
\begin{aligned}
& d\{y(t)P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))]\} \\
= \ & P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))]dy \\
& +y(t)\{P_t(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))] \\
& +P(t,\alpha(t))[(\lambda-\hat{z})H_t(t,\alpha(t))+R_t(t,\alpha(t))]\}dt \\
& +\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)y(t)[(\lambda-\hat{z})H(t,j)+R(t,j)]dt+\{\cdots\}dW(t) \\
= \ & P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))] \\
& [r(t,\alpha(t))y(t)+B(t,\alpha(t))\pi(t)+f(t,\alpha(t))]dt \\
& +y(t)\{[[\rho(t,\alpha(t))-2r(t,\alpha(t))]P(t,\alpha(t))-\xi \\
& -\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)][(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))] \\
& +P(t,\alpha(t))[(\lambda-\hat{z})[[r(t,\alpha(t))+\tfrac{\xi}{P(t,\alpha(t))}]H(t,\alpha(t)) \\
& -\tfrac{1}{P(t,\alpha(t))}\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,\alpha(t))]] \\
& +[[r(t,\alpha(t))+\tfrac{\xi}{P(t,\alpha(t))}]R(t,\alpha(t))-f(t,\alpha(t)) \\
& -\tfrac{1}{P(t,\alpha(t))}\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[R(t,j)-R(t,\alpha(t))]]\}dt \\
& +\textstyle\sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)y(t)[(\lambda-\hat{z})H(t,j)+R(t,j)]dt+\{\cdots\}dW(t) \\
= \ & P(t,\alpha(t))\{y(t)[\rho(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))]-f(t,\alpha(t))] \\
& +[(\lambda-\hat{z})H(t,\alpha(t))+R(t,\alpha(t))][B(t,\alpha(t))\pi(t)+f(t,\alpha(t))]\}dt \\
& +\{\cdots\}dW(t).
\end{aligned}
\tag{4.38}
$$

Integrating the above equation and (4.26) from 0 to T, taking expectations and adding them together, we get

$$
\begin{aligned}
& \tfrac{1}{2}(1-\xi)Ey^2(T) + (1-\xi)(\lambda-\hat{z})Ey(T) \\
= \; & E\int_0^T \{\tfrac{1}{2}[\rho(t,\alpha(t))P(t,\alpha(t)) - \xi]y^2(t) \\
& + P(t,\alpha(t))y(t)[B(t,\alpha(t))\pi(t) + f(t,\alpha(t))] \\
& + \tfrac{1}{2}P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t)\}dt \\
& + E\int_0^T P(t,\alpha(t))\{y(t)[\rho(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))] - f(t,\alpha(t))] \\
& + [(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))][B(t,\alpha(t))\pi(t) + f(t,\alpha(t))]\}dt \\
= \; & E\int_0^T \{\tfrac{1}{2}P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t) \\
& + \big[P(t,\alpha(t))[y(t) + (\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))]\big]B(t,\alpha(t))\pi(t) \\
& + \tfrac{1}{2}[\rho(t,\alpha(t))P(t,\alpha(t)) - \xi]y^2(t) \\
& + \rho(t,\alpha(t))P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))]y(t) \\
& + f(t,\alpha(t))P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))]\}dt \\
= \; & E\int_0^T \tfrac{1}{2}P(t,\alpha(t))[\pi(t) - \pi^*_\lambda(t,y(t),\alpha(t))]' \\
& [\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][\pi(t) - \pi^*_\lambda(t,y(t),\alpha(t))]dt \\
& + E\int_0^T \{f(t,\alpha(t))P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))] - \tfrac{\xi}{2}y^2(t) \\
& - \tfrac{1}{2}\rho(t,\alpha(t))P(t,\alpha(t))[(\lambda-\hat{z})H(t,\alpha(t)) + R(t,\alpha(t))]^2\}dt,
\end{aligned}
$$

where $\pi^*_\lambda(t,y,i)$ is defined as the right-hand side of (4.33).

Consequently,

$$
\begin{aligned}
&J_2(y_0, i_0, \pi_\lambda(\cdot), \lambda) \\
= \;& E\int_0^T \tfrac{1}{2} P(t,\alpha(t))[\pi(t) - \pi_\lambda^*(t, y(t), \alpha(t))]' \\
& [\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][\pi(t) - \pi_\lambda^*(t, y(t), \alpha(t))]dt \\
& + E\int_0^T \tfrac{1}{2} P(t,\alpha(t))[\pi(t) - \pi_\lambda^*(t, y(t), \alpha(t))]' \\
& [\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][\pi(t) - \pi_\lambda^*(t, y(t), \alpha(t))]dt \\
& + E\int_0^T \{ f(t,\alpha(t))P(t,\alpha(t))[(\lambda - \hat{z})H(t,\alpha(t)) + R(t,\alpha(t))] \\
& - \tfrac{1}{2}\rho(t,\alpha(t))P(t,\alpha(t))[(\lambda - \hat{z})H(t,\alpha(t)) + R(t,\alpha(t))]^2\}dt \\
& - \tfrac{1-\xi}{2}[2\hat{z}(\lambda - \hat{z}) + \hat{z}^2] \\
= \;& \int_0^T \tfrac{1}{2} \textstyle\sum_{i=1}^l p_{i_0 i}(t) P(t,i)[\pi(t) - \pi_\lambda^*(t,y,i)]' \\
& [\sigma(t,i)\sigma(t,i)'][\pi(t) - \pi_\lambda^*(t,y,i)]dt \\
& + \int_0^T \textstyle\sum_{i=1}^l p_{i_0 i}(t)\{ f(t,i)P(t,i)[(\lambda - \hat{z})H(t,i) + R(t,i)] \\
& - \tfrac{1}{2}\rho(t,i)P(t,i)[(\lambda - \hat{z})H(t,i) + R(t,i)]^2\}dt \\
& - \tfrac{1-\xi}{2}[2\hat{z}(\lambda - \hat{z}) + \hat{z}^2] \\
= \;& \int_0^T \tfrac{1}{2} \textstyle\sum_{i=1}^l p_{i_0 i}(t) P(t,i)[\pi(t) - \pi_\lambda^*(t,y,i)]' \\
& [\sigma(t,i)\sigma(t,i)'][\pi(t) - \pi_\lambda^*(t,y,i)]dt \\
& + \theta_4(\lambda - \hat{z})^2 + [\theta_5 - (1-\xi)\hat{z}](\lambda - \hat{z}) + \theta_6 - \tfrac{1-\xi}{2}\hat{z}^2.
\end{aligned}
$$

Since $P(t,i) > 0$ by Proposition 4.2, it follows immediately that the optimal feedback control is given by (4.33) and the optimal value is given by (4.34), provided that the corresponding equation (4.12) under the feedback control (4.33) has a solution. Using the similar argument as that in the proof of Proposition 4.3, we can finish the remaining proof of this theorem. □

Theorem 4.2 *Problem (4.10) has an optimal feedback control*

$$\pi_2^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x - x_0e^{\mu t} + (\lambda^* - (z - e^{\mu T}x_0))H(t,i) + R(t,i)], \tag{4.39}$$

where

$$\lambda^* - (z - e^{\mu T}x_0) = -\frac{\theta_5 - (1-\xi)(z - e^{\mu T}x_0)}{2\theta_4}, \tag{4.40}$$

and the corresponding optimal value is

$$J_2(x_0, i_0, \pi_2^*(\cdot)) \ = \theta_6 - \tfrac{1-\xi}{2}(z - e^{\mu T}x_0)^2 - \tfrac{[\theta_5 - (1-\xi)(z - e^{\mu T}x_0)]^2}{4\theta_4}. \tag{4.41}$$

Proof. We can use the similar argument as that in the proof of Proposition 3.3. Since $\theta_4 < 0$, we maximize the quadratic function (4.34) over $\lambda - \hat{z}$ and conclude that the maximizer is given by

$$\lambda^* - \hat{z} = -\frac{\theta_5 - (1-\xi)\hat{z}}{2\theta_4},$$

whereas the optimal value and control law are obtained by

$$J_2(y_0, i_0, \pi_2^*(\cdot)) \ = \theta_6 - \tfrac{1-\xi}{2}\hat{z}^2 - \tfrac{[\theta_5 - (1-\xi)\hat{z}]^2}{4\theta_4} \ ,$$

and

$$\pi_2^*(t,y,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[y + (\lambda^* - \hat{z})H(t,i) + R(t,i)].$$

Replacing $y = x - x_0 e^{\mu t}$ and $\hat{z} = z - e^{\mu T}x_0$ in the above results, we get (4.41) and (4.39) respectively, with (4.40). □

Theorem 4.3 *We have*

$$1 - \xi + 2\theta_4 > 0. \tag{4.42}$$

Moreover, the minimum optimal value of (4.10) over $z \in \mathbb{R}$ is

$$J^*_{2\_min} = \theta_6 - \frac{\theta_5^2}{2(1-\xi+2\theta_4)}, \tag{4.43}$$

with the corresponding expected terminal wealth

$$z_{2\_min} := \frac{\theta_5}{1-\xi+2\theta_4} + e^{\mu T}x_0, \tag{4.44}$$

and the corresponding Lagrange multiplier $\lambda^*_{min} = 0$. Furthermore, the portfolio that achieves the above minimum optimal value is

$$\pi^*_{2\_min}(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x - e^{\mu t}x_0 - (z - e^{\mu T}x_0)H(t,i) + R(t,i)]. \tag{4.45}$$

Proof. Similarly to Theorem 3.3, we can prove this theorem. □

4.4 Solution to Model III with Regime Switching

In this section we study Model III for the market with regime switching.

Consider the following three systems of ODEs:

$$\begin{cases} \dot{P}(t,i) = [\rho(t,i) - 2r(t,i)]P(t,i) - \sum_{j=1}^{l} q_{ij}P(t,j) - 1 \\ P(T,i) = 0, \ \ i = 1, 2, \cdots, l, \end{cases} \tag{4.46}$$

$$\begin{cases} \dot{g}_1(t,i) = [\rho(t,i) - r(t,i)]g_1(t,i) - \sum_{j=1}^{l} q_{ij}g_1(t,j) \\ g_1(T,i) = 1, \ \ i = 1, 2, \cdots, l, \end{cases} \tag{4.47}$$

$$\begin{cases} \dot{g}_2(t,i) = [\rho(t,i) - r(t,i)]g_2(t,i) - \sum_{j=1}^{l} q_{ij}g_2(t,j) - f(t,i)P(t,i) \\ g_2(T,i) = 0, \ \ i = 1, 2, \cdots, l. \end{cases} \tag{4.48}$$

Proposition 4.5 *For each fixed $\hat{\lambda} \in \mathbb{R}$, problem (4.17) has an optimal feedback control*

$$\pi^*_{\hat{\lambda}}(t,y,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[y + \frac{\hat{\lambda}g_1(t,i) + g_2(t,i)}{P(t,i)}], \tag{4.49}$$

and the corresponding optimal value is

$$\tilde{J}_3(y_0, i_0, \pi^*_{\hat{\lambda}}(\cdot)) = \theta_4'\hat{\lambda}^2 + [\theta_5' - \hat{z}]\hat{\lambda} + \theta_6', \tag{4.50}$$

where

$$\theta_4' = -\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)\frac{\rho(t,i)g_1^2(t,i)}{2P(t,i)}dt < 0, \tag{4.51}$$

$$\theta_5' = \int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)g_1(t,i)[f(t,i) - \frac{\rho(t,i)g_2(t,i)}{P(t,i)}]dt, \tag{4.52}$$

$$\theta_6' = \int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)g_2(t,i)[f(t,i) - \frac{\rho(t,i)g_2(t,i)}{2P(t,i)}]dt. \tag{4.53}$$

Proof. Let $(y(\cdot), \pi(\cdot))$ be any given admissible pair. Applying the Ito formula to $\phi_3(t,y,i) = \frac{1}{2}P(t,i)y^2$ and $\phi_6(t,y,i) = [\hat{\lambda}g_1(t,i) + g_2(t,i)]y$, we obtain (4.26), and

$$\begin{aligned}
& d\{[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))]y(t)\} \\
= \;& \{[[\rho(t,\alpha(t)) - r(t,\alpha(t))][\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))] \\
& -f(t,\alpha(t))P(t,\alpha(t)) - \textstyle\sum_{i=1}^{l} q_{\alpha(t)j}(t)[\hat{\lambda}g_1(t,j) + g_2(t,j)]]y(t) \\
& +[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))][r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t) + f(t,\alpha(t))] \\
& +\textstyle\sum_{i=1}^{l} q_{\alpha(t)j}(t)[\hat{\lambda}g_1(t,j) + g_2(t,j)]y(t)\}dt + \{\cdots\}dW(t) \\
= \;& \{[\rho(t)[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))] - f(t,\alpha(t))P(t,\alpha(t))]y(t) \\
& +[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))][B(t,\alpha(t))\pi(t) + f(t,\alpha(t))]\}dt + \{\cdots\}dW(t).
\end{aligned} \tag{4.54}$$

Integrating the above equation and (4.26) from 0 to T, taking expectations and adding them together, we get

$$\begin{aligned}
& \hat{\lambda}Ey(T) \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t) \\
& +[y(t) + \hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))]B(t,\alpha(t))\pi(t)\}dt \\
& +E\int_0^T \{\tfrac{1}{2}[\rho(t,\alpha(t))P(t,\alpha(t)) - 1]y^2(t) \\
& +[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))][\rho(t,\alpha(t))y(t) + f(t,\alpha(t))]\}dt \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t,\alpha(t))[\pi(t) - \pi^*_{\hat{\lambda}}(t,y(t),\alpha(t))]' \\
& [\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][\pi(t) - \pi^*_{\hat{\lambda}}(t,y(t),\alpha(t))]\}dt \\
& +E\int_0^T \{f(t,\alpha(t))[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))] \\
& -\tfrac{1}{2}y^2(t) - \tfrac{\rho(t,\alpha(t))[\hat{\lambda}g_1(t,\alpha(t)) + g_2(t,\alpha(t))]^2}{2P(t,\alpha(t))}\}dt
\end{aligned}$$

where $\pi^*_{\hat{\lambda}}(t,y,i)$ is defined as the right-hand side of (4.49). Consequently,

$$
\begin{aligned}
&\tilde{J}_3(y_0, i_0, \pi_{\hat{\lambda}}(\cdot), \hat{\lambda}) \\
= \;& E\int_0^T \{\tfrac{1}{2}P(t,\alpha(t))[\pi(t)-\pi^*_{\hat{\lambda}}(t,y(t),\alpha(t))]' \\
& [\sigma(t,\alpha(t))\sigma(t,\alpha(t))'][\pi(t)-\pi^*_{\hat{\lambda}}(t,y(t),\alpha(t))]\}dt \\
& +E\int_0^T \{f(t)[\hat{\lambda}g_1(t,\alpha(t))+g_2(t,\alpha(t))] - \tfrac{\rho(t,\alpha(t))[\hat{\lambda}g_1(t,\alpha(t))+g_2(t,\alpha(t))]^2}{2P(t,\alpha(t))}\}dt - \hat{\lambda}\hat{z} \\
= \;& \int_0^T \{\tfrac{1}{2}\sum_{i=1}^{l} p_{i_0 i}(t)P(t,i)[\pi(t)-\pi^*_{\hat{\lambda}}(t,y,i)]'[\sigma(t,i)\sigma(t,i)'][\pi(t)-\pi^*_{\hat{\lambda}}(t,y,i)]\}dt \\
& +[-\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)\tfrac{\rho(t,i)g_1^2(t,i)}{2P(t,i)}dt]\hat{\lambda}^2 \\
& +\{\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)g_1(t,i)[f(t,i)-\tfrac{\rho(t,i)g_2(t,i)}{P(t,i)}]dt - \hat{z}\}\hat{\lambda} \\
& +\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)g_2(t,i)[f(t,i)-\tfrac{\rho(t,i)g_2(t,i)}{2P(t,i)}]dt.
\end{aligned}
$$

Using the similar argument as that in the proof of Theorem 4.2, we complete the proof. □

Theorem 4.4 *Problem (2.9) has an optimal feedback control*

$$\pi_3^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1}B(t,i)'[x - e^{\mu t}x_0 + \frac{\hat{\lambda}^* g_1(t,i)+g_2(t,i)}{P(t,i)}] \quad (4.55)$$

where

$$
\begin{aligned}
\hat{\lambda}^* = \;& -\frac{\theta_5' - (z - e^{\mu T}x_0)}{2\theta_4'} \\
= \;& \frac{\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)g_1(t,i)[f(t,i)-\frac{\rho(t,i)g_2(t,i)}{P(t,i)}]dt - (z-e^{\mu T}x_0)}{\int_0^T \sum_{i=1}^{l} p_{i_0 i}(t)\frac{\rho(t,i)g_1^2(t,i)}{P(t,i)}dt}
\end{aligned}
\quad (4.56)
$$

and the corresponding optimal value is

$$J_3(x_0, i_0, \pi_3^*(\cdot)) = \theta_6' - \frac{[\theta_5' - (z - e^{\mu T}x_0)]^2}{4\theta_4'}. \quad (4.57)$$

Proof. We can use the similar argument in the proof of Proposition 3.3. Since $\theta_4' < 0$, we maximize the quadratic function (4.50) over $\hat{\lambda} - \hat{z}$ and conclude that the maximizer is given by

$$\hat{\lambda}^* = -\frac{\theta_5' - \hat{z}}{2\theta_4'} = \frac{\int_0^T \sum_{i=1}^l p_{i_0 i}(t) g_1(t,i)[f(t,i) - \frac{\rho(t,i) g_2(t,i)}{P(t,i)}]dt - \hat{z}}{\int_0^T \sum_{i=1}^l p_{i_0 i}(t) \frac{\rho(t,i) g_1^2(t,i)}{P(t,i)} dt},$$

whereas the optimal value and control law are obtained by

$$J_3(y_0, i_0, \pi_3^*(\cdot)) = \theta_6' - \frac{[\theta_5' - \hat{z}]^2}{4\theta_4'},$$

and

$$\pi_3^*(t,y,i) = -[\sigma(t,i)\sigma(t,i)']^{-1} B(t,i)'[y + \frac{\hat{\lambda}^* g_1(t,i) + g_2(t,i)}{P(t,i)}].$$

Replacing $y = x - x_0 e^{\mu t}$ and $\hat{z} = z - e^{\mu T} x_0$ in the above results, we get (4.57) and (4.55) respectively, with (4.56). □

Similarly to Theorem 3.6, we have the following result.

Theorem 4.5 *The minimum optimal value of (4.11) over $z \in \mathbb{R}$ is*

$$J_{3\_min}^* = \theta_6', \tag{4.58}$$

with the corresponding expected terminal wealth

$$z_{3\_min} := \theta_5' + e^{\mu T} x_0, \tag{4.59}$$

and the corresponding Lagrange multiplier $\hat{\lambda}_{min}^* = 0$. Moreover, the portfolio that achieves the above minimum optimal value is

$$\pi_{3\_min}^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)']^{-1} B(t,i)'[x - e^{\mu t} x_0 + g_2(t,i)]. \tag{4.60}$$

Proof. Similarly to Theorem 3.6, we can prove this theorem. □

Chapter 5

Conclusion

In this thesis we have investigated the problem of tracking a deterministic, continuously compounding growth by composing appropriate portfolios in financial markets. We have solved several models completely in two different market environments, one with deterministic market parameters, and the other with regime switching. In Chapter 2, we presented the general models and assumptions of the whole thesis. Then in Chapter 3 and Chapter 4, we investigated the tracking models in the market with deterministic parameters and the one with regime switching, respectively. It should be noted that, although the latter is more general which includes the former as a special case, it is important to start with a simple one so as to catch the essence of the problem. Also, while the whole work is in general based on the stochastic LQ approach, there are subtle details in each model and each particular market. Due to the terminal return constraint, we used Lagrangian approach to tackle Model II and Model III. On the other hand, when the market has regime switching, we employed a Markov-chain modulated diffusion formulation to study the problem.

The models under investigation in this thesis feature no transaction cost, short-selling is allowed and wealth constraint is absent. For the future work, we may consider cases with constrained portfolios, such as the one with prohibition of short-selling (i.e., there is a portfolio constraint $\pi(t) \geq 0$), with prohibition of bankruptcy (i.e., there is a wealth constraint $x(t) \geq 0$), or with transaction cost. Another interesting case is when all the market parameters are general random coefficients, instead of Markov-modulated. All these remain interesting, albeit challenging, research problems.

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